From the Editor

Divisors, maps, and interest

It isn't often that I get a response from my students – a yawn, the occasional snore, not much else. So I was a bit taken aback when David came to see me after a lecture to ask about perfect numbers. A *perfect number* is a positive integer which is the sum of its positive divisors, excluding the number itself. Thus the first four perfect numbers are

$$\begin{aligned} 6 &= 1 + 2 + 3, \\ 28 &= 1 + 2 + 4 + 7 + 14, \\ 496 &= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248, \\ 8128 &= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064. \end{aligned}$$

What David did was to multiply these divisors instead of adding them, i.e.

$$\begin{aligned} 1 \times 2 \times 3 &= 6, \\ 1 \times 2 \times 4 \times 7 \times 14 &= 784 = 28^2. \end{aligned}$$

Similarly, the product of the divisors of 496 is 496^4 and the product of the divisors of 8128 is 8128^6. David wanted to know whether this product is always a power of the number. A good question!

Coincidentally, a letter arrived from a reader, V. Tyagi of Shyam Lal College, Delhi University, posing the same problem, but with no reference to perfect numbers. He asked for a formula for the product of the positive divisors of a number.

Perfect numbers turn out to be a red herring. Start with any positive integer n; take $n = 24$ as an example. Its positive divisors pair off as follows so that their product is 24:

$$24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6.$$

Thus, the product of the positive divisors of 24, including 24 itself, is 24^4, and the power is half the number of positive divisors of 24. This works for all numbers which are not perfect squares. But take 36, for example, a perfect square. This time we obtain

$$36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6,$$

and the product of the positive divisors of 36 is $36^4 \times 6$ (6 can only be included in the product once), or $36^{9/2}$. Again the power is half the number of positive divisors; this works for all perfect squares.

It remains to work out how many positive divisors a number n has. First factorize n into its prime factors, say

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

(for example $24 = 2^3 \times 3^1$ or $36 = 2^2 \times 3^2$). The positive divisors of n are the numbers

$$p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r},$$

where $l_i = 0, 1, \ldots, k_i$ for $i = 1, \ldots, r$, so there are $k_i + 1$ possibilities for l_i and the number of positive divisors of n is

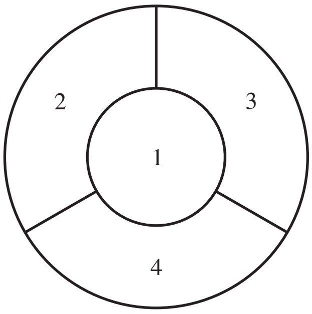
$$(k_1 + 1)(k_2 + 1) \cdots (k_r + 1).$$

For example, 24 has $2 \times 2 = 8$ positive divisors, and 36 has $3 \times 3 = 9$. Hence, the product of the positive divisors of n is

$$n^{\frac{1}{2}(k_1+1)(k_2+1)\cdots(k_r+1)}.$$

Simple, but pretty!

Another of my students, Lloyd, found an interesting item on the web. (This sudden bout of interest must be catching!) We had been talking about the famous four-colour theorem. Given any map, how many colours are needed to colour the countries so that every two countries with a common boundary are coloured differently? The map shown in the figure below needs four colours, labelled 1 to 4; you can even colour the outside using colour 1 again.



(I have to insist that students do not literally colour the maps because I am colour-blind. Strange for a colour-blind person to be teaching map-colouring!) It was conjectured way back in the 19th century that four colours suffice for any map. This was finally proved in 1976 by Ken Appel and Wolfgang Haken at the University of Illinois, but their proof requres the checking of a large number of cases which can only realistically be done by computer. How do we know that the computer got it right? Now Georges Gouthier at Microsoft's research laboratory in Cambridge, UK, and Benjamin Werner at INRIA in France have translated the proof in to a language called *Coq*, used to represent logical propositions, and have created logic-checking software to check the proof. I quote from the *New Scientist* website, but more information can be found on links there, including Georges Gouthier's website. Maybe someone will explain it to me sometime.

To bring us down to earth, one of out readers, whose anonymity had better be preserved to avoid begging letters, has written about his monthly savings scheme, in which he saves £250 per month at a rate of interest of 7% per annum. The bank provided table 1 showing the interest earned after a year using a very precise way of calculating depending on the number of days in each month.

Table 1

Period of calculation		Number of	Cleared account	Interest	Interest
From	To	days	balance	rate	earned
04/04/2005	03/05/2005	30	£250	7.00%	£1.44
04/05/2005	03/06/2005	31	£500	7.00%	£2.97
04/06/2005	03/07/2005	30	£750	7.00%	£4.32
04/07/2005	03/08/2005	31	£1000	7.00%	£5.95
04/08/2005	03/09/2005	31	£1250	7.00%	£7.43
04/09/2005	03/10/2005	30	£1500	7.00%	£8.63
04/10/2005	03/11/2005	31	£1750	7.00%	£10.40
04/11/2005	03/12/2005	30	£2000	7.00%	£11.51
04/12/2005	03/01/2006	31	£2250	7.00%	£13.38
04/01/2006	03/02/2006	31	£2500	7.00%	£14.86
04/02/2006	03/03/2006	28	£2750	7.00%	£14.77
04/03/2006	03/04/2006	31	£3000	7.00%	£17.84
				Total	£113.49

He was told that this was equivalent to an annual equivalent rate (AER) of interest of 7.07%. Being a curious sort of guy, he asked the bank where this figure came from. After three letters from the bank, he gave up and asked us. Our best attempt is to say that the first payment after a year becomes

$$£250\left(1+\tfrac{7}{100}\right).$$

The second payment, using the compound interest formula, becomes

$$£250\left(1+\tfrac{7}{100}\right)^{335/365},$$

the third

$$£250\left(1+\tfrac{7}{100}\right)^{304/365},$$

and so on. Adding up the twelve figures, I got £3113.27, 21 pence out. Given possible calculator errors on my part, is this close enough? And is it right?

Finally, another reader, Mr Sastry of Bangalore, India, has sent us a couple of problems which you might like to try.

1. A Pythagorean triangle, i.e. a right-angled triangle with integer sides, has area 451 350. What is its semi-perimeter?

2. The sides of a triangle are 6324, 7493, and 9805. What is the radius of its incircle?

As a clue, both problems are connected with the year. Happy solving!