Some Group Elements Defined by Commutators

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Grün’s lemma is our starting-point. This elegant result (Grün (1935)) states that the group \( G \) contains \( \delta(G) \) properly if \( \zeta_2(G) \) contains \( \zeta(G) \) properly. Here \( \delta(G) \) is the subgroup of \( G \) generated by all elements of the form \( x^{-1}y^{-1}xy \), which we shall write as \([x, y]\); while \( \zeta(G/\zeta(G)) = \zeta_2(G)/\zeta(G) \) defines the subgroup \( \zeta_2(G) \) of \( G \).

One proof of Grün’s lemma depends on the mapping \( \mu_a: G \rightarrow G \) defined by

\[
\mu_a(x) = [a, x] \quad (x \in G) \tag{1}
\]

where \( a \) is a fixed element of \( \zeta_2(G) \) not in \( \zeta(G) \). For \([a, x] \in \zeta(G)\); and it follows that \( \mu_a \) is a homomorphism with non-trivial abelian image, so that the kernel of \( \mu_a \) is a proper subgroup of \( G \) containing \( \delta(G) \).

Hence the question: for which elements \( b \) of \( G \) is \( \mu_b \), defined in (1), a homomorphism? We have seen that every element of \( \zeta_2(G) \) has this property. The complete answer is given in the following theorem.

**Theorem 1.** The mapping \( \mu_a: G \rightarrow G \) defined by (1) is a homomorphism if and only if \( a \in \zeta_2(G) \).

**Proof.** If \( a \) is such that \( \mu_a \) is a homomorphism then we have, for all \( x \) and \( y \) in \( G \),

\[
[a, xy] = [a, x][a, y]. \tag{2}
\]

At this point we have to introduce the standard commutator identity

\[
[u, vw] = [u, w][u, v]^w, \tag{3}
\]

in which \( x^y \) is an abbreviation for \( y^{-1}xy \). Now put \( y = a \) in (2) and apply (3). Thus \([a, x]^a = [a, x] \) for all \( x \), and since \([a, x] = a^{-1}a^x \) we see that \( a^xa = a^xa^x \); that is, \( a \) commutes with all its conjugates.

We can rewrite (2) as \( a^{-1}a^{xy} = a^{-1}a^xa^{-1}a^x \), and deduce that \( aa^{-x}a^{-y}a^{xy} = 1 \). Now consider \([a, x, y] \), by which we mean \([a, x, y] \). We have \([a, x, y] = a^{-x}aa^{-y}a^{xy} \), and in our particular case we find that \([a, x, y] = 1 \). Thus \([a, x] \in \zeta(G) \), and \( a \in \zeta_2(G) \).

Theorem 1 generalises a result of Levi (1942) which is conveniently to be found in the textbook of Kurosh (1956), Vol. I, p. 101, along with remarks useful to those inexperienced in commutator manipulation. The result states that (2) holds for all \( a, x \) and \( y \) in \( G \) if and only if \( G \) is nilpotent of class 2 (which is equivalent to: \( G = \zeta_2(G) \)).

We leave to the reader the proof of the rather easier result that \( b \in \zeta_2(G) \) if and only if for every \( x, y \) in \( G \)

\[
[xy, b] = [x, b][y, b]. \tag{4}
\]
Now (2) can be regarded as a sort of distributive law for commutation and multiplication, and Levi considers an associative law in the same spirit, namely

$$[[x, y], z] = [x, [y, z]].$$  \hspace{1cm} (5)

See Kurosh (1956), Vol. I, pp. 99–100, and Robinson (1972), Vol. 2, p. 46; (Chapter 7 of the latter contains some proofs by the sort of calculation we give below). We shall generalise Levi's result about (5); first a lemma, in which \(C(\delta(G))\) stands for the centralizer of \(\delta(G)\).

**Lemma 2.** Let \(a\) be a fixed element in the group \(G\). Then \([a, x, y] = [a, y, x]\) for all \(x, y\) in \(G\) if and only if \(a \in C(\delta(G))\).

**Proof.** Suppose that \([a, x, y] = [a, y, x]\). Put \(y = a\) and obtain \([a, x, a] = 1\), which as in the proof of Theorem 1 implies that \(a\) and \(a^x\) commute. The hypothesis gives \(a^{-x}aa^{-x}a^{y} = a^{-x}aa^{-x}a^{x}\). Thus we have \(a^{xy} = a^{sx}\), so \(a\) commutes with \(xy^{-1}x^{-1}\), and \(a \in C(\delta(G))\).

Conversely, if \(a \in C(\delta(G))\) then \([x, y, a] = 1\) for all \(x, y\), and the case \(x = a\) gives \([a^x, a] = 1\). The steps in the previous paragraph can thus be reversed, with the result that \([a, x, y] = [a, y, x]\).

The following consequence of Lemma 2, though aside from the theme of this note, is of interest.

**Corollary.** Let \(a, b\) be fixed elements in the group \(G\). Then \([a, b] \in \zeta(\delta(G))\) if and only if \([a, b, x, y] = [a, b, y, x]\) for all \(x, y \in G\).

**Theorem 3.** Let \(a\) be a fixed element in the group \(G\). Then \([(x, a, y] = [x, [a, y]]\) for all \(x, y\) in \(G\) if and only if \(a \in C(\delta(G))\).

**Proof.** If \([(x, a, y] = [x, [a, y]]\) then

$$[x, a, y][a, y, x] = 1,$$  \hspace{1cm} (6)

and with \(x = a\) we soon find that \([a^x, a] = 1\) for all \(y\). So \([x, a, y] = [a, x, y]^{-1}\), and Lemma 2 at once gives \(a \in C(\delta(G))\). The converse is clear.

Levi’s result is a corollary: (5) holds if and only if \(G = C(\delta(G))\), which is another way of saying that \(G\) is nilpotent of class 2. This can also be generalised in a rather different way.

**Theorem 4.** Let \(a\) be a fixed element in the group \(G\). Then \([a, x, y] = [a, [x, y]]\) for all \(x, y\) in \(G\) if and only if \(a \in \zeta_2(G)\).

**Proof.** If \([a, x, y] = [a, [x, y]]\) then

$$[a, x, y][x, y, a] = 1.$$  \hspace{1cm} (7)

From the case \(x = a\) follows \([a^x, a] = 1\).

At this point we appeal to another standard identity:

$$[w, u, v^w][v, w, u^w][u, v, w^u] = 1.$$  \hspace{1cm} (8)
In particular we have in $G$

$$[x, a, y^a][y, x, a^x][a, y, x^a] = 1. \quad (9)$$

Now $[a, y, x^a] = [a, y, x]$ because $x^a = x[x, a]$ and $a$ commutes with its conjugates; if in doubt apply (3) to $[[a, y], x[x, a]]$. Further $[y, x, a^x] = [a^x, y, x]^{-1}$ by (7), and

$$[a^x, y] = (a, y)^x = [a, y][a, y, y] = [a, y]$$

since (7) with $x = y$ gives $[a, y, y] = 1$. So (9) becomes $[x, a, y^a] = 1$. Since $y$ is arbitrary this implies $[x, a] \in \zeta(G)$.

The reader may care to prove for himself a similar result featuring

$$[[x, y], a] = [x, [y, a]]. \quad (10)$$

References