OPTIMAL CONSTRAINED CONFIDENCE ESTIMATION OF THE POISSON MEAN VIA TAIL FUNCTIONS

BOREK PUZA ∗ ∗ ∗ AND MO YANG,∗ Australian National University

Abstract

A methodology is proposed for exact confidence estimation of the Poisson mean when that parameter is constrained. It is shown how tail functions can be used to construct a suitable confidence interval when the mean is bounded above, below, or both, and how this interval can be engineered for optimality in terms of prior expected length and other criteria. The theory is illustrated by way of comparison with the unified approach of Feldman and Cousins (1998).

Keywords: Poisson mean; confidence interval; constrained inference; tail function; unified approach

2010 Mathematics Subject Classification: Primary 62F25 Secondary 62F30

1. Introduction

When estimating a Poisson mean it is often desirable to construct an exact confidence interval (CI) for that parameter, where exact means that the parameter lies inside the CI (considered as a random variable) with a specified minimum probability (e.g. 95%) for all possible values of the parameter. A formula for such an exact CI was developed by Garwood (1936); this formula involves quantiles of the $\chi^2$ distribution and is detailed in Section 4.

Sometimes the Poisson mean is constrained in a particular way. For example, there may be a background Poisson process in play with a known mean which implies a lower bound for the overall mean in question. An example was given by Mandelkern (2002) in relation to the Karlsruhe Rutherford Medium Energy Neutrino (KARMEN) experiment. This experiment involved inference on the mean of a total number of subatomic events, comprising a count for the appearance reaction of interest and a count for the background process with mean 2.88, where both counts were believed to follow independent Poisson processes. Conversely, an upper bound applies if the observable Poisson count corresponds to a subcategory of a broader count whose mean is known. In some situations the Poisson mean may be bounded in both ways.

For sufficiently ‘extreme’ values of the Poisson count, the exact CI will contain some implausible values (e.g. values much larger than the upper bound). So long as the CI contains some plausible values, an exact CI can be reported following truncation, meaning the elimination of all implausible values. However, in some cases the overall Poisson count is so extreme that the truncated CI contains no plausible values at all (e.g. lies entirely above the upper bound).
Such cases lead to the awkward result of an empty CI and motivate the need for an alternative formula that produces a CI which is guaranteed to be exact and nonempty after truncation.

Several suggestions have been made for constructing such an interval. For example, the Bayesian approach is considered in Roe and Woodroofe (2000) and Alvarez (2007). However, this approach is not exact by virtue of the fact that it leads to constrained CIs for the Poisson mean that do not have at least the desired frequentist coverage probability for all values of that mean.

Another suggestion is to apply the unified approach of Feldman and Cousins (1998), which does provide an exact constrained CI for the Poisson mean and is guaranteed to be nonempty. However, this approach has several drawbacks. Firstly, it sometimes results not in a single interval but rather a confidence region made up of two separate intervals, which is somewhat counterintuitive. This awkward feature can be dealt with by reporting the single interval formed by the lowest and highest values in the entire disjointed confidence region. Although the result is an exact CI, this 'fix' could be criticized for being overly conservative (i.e. producing an unnecessarily wide interval). Secondly, the unified approach leads to CIs whose bounds sometimes move in counterintuitive directions relative to changes in the lower and upper bounds for the Poisson mean. Thirdly, the unified approach is inflexible and may not be ‘best’ in every situation. Several other problems with the unified approach can be found in Zech (2008). For a broader discussion of the unified and some other approaches to constrained interval estimation, the reader is referred to Mandelkern (2002).

Yet another suggestion is to apply the tail functions approach of Puza and O’Neill (2005), (2006a), (2006b), (2008), (2009). This leads to what may be termed the generalized Garwood CI. As will be seen, certain classes of tail function can be used to produce a suitable exact CI for the Poisson mean when that mean is bounded below, above, or both, and this can be done in a way which avoids the two awkward features noted above in relation to the unified approach.

Another advantage of the generalised Garwood CI is that it can be engineered to provide confidence estimation that is optimal in some meaningful sense. For example, suppose that prior information is available, and we do not wish to apply the Bayesian approach because of issues with its coverage probabilities. Then that prior information can be used to construct an exact CI that has the smallest possible prior expected length amongst all the CIs in a wide class. In many cases a CI can be found which is significantly shorter on average than the CI implied by the unified approach.

The unified approach to interval estimation for a constrained Poisson mean is reviewed in Section 2. Section 3 defines the tail functions approach and shows how it can be used to generalize the Garwood CI. In Section 4, CIs obtained by the unified and tail functions approaches are compared by way of an example where the Poisson mean is bounded above. Thereby it is shown how the tail functions approach can lead to a CI that is shorter on average than the CI implied by the unified approach. Section 5 contains a summary and discussion, and a key result is proved in Appendix A.

2. The unified approach

Consider the problem of constructing a $1 - \alpha$ CI for a mean $\lambda$ based on $X$ events under the model $(X | \lambda) \sim \text{Poisson}(\lambda)$, where the CI is required to be ‘exact’ (i.e. have a frequentist coverage probability of at least $1 - \alpha$ for all $\lambda$) and $\lambda$ is constrained to be in a specified interval $[a, b]$ (where $a$ could be zero and $b$ could be infinity).
The method of Feldman and Cousins (1998) may be applied in this context to define the $1 - \alpha$ unified approach confidence interval (UACI) for $\lambda$, as follows. Consider the likelihood ratio

$$R_\lambda(x) = \frac{f(x \mid \lambda)}{f(x \mid \hat{\lambda})},$$

where $f(x \mid \lambda) = e^{-\lambda} \lambda^x / x!$ is the Poisson probability mass function with mean $\lambda$, and $\hat{\lambda}$ is the maximum likelihood estimate of $\lambda$ under the constraint. For example, if $a > 0$ and $b = \infty$ then $\hat{\lambda} = a \mathbf{1}_{[x<a]} + x \mathbf{1}_{[x\geq a]}$, where $\mathbf{1}_E$ is the standard indicator function for event $E$.

The unified approach requires that for each $\lambda > 0$ we find the largest real number $k$ and the corresponding two integers $A(\lambda)$ and $B(\lambda)$ such that $R_\lambda(x) \geq k$ for all $x \in [A(\lambda), B(\lambda)]$ and $P(A(\lambda) \leq X \leq B(\lambda) \mid \lambda) \geq 1 - \alpha$.

Inverting $x = A(\lambda)$ and $x = B(\lambda)$ then yields the upper and lower bounds, respectively, of the $1 - \alpha$ UACI for $\lambda$, $[L(x), U(x)]$. One issue with this approach (amongst several) is that the functions $x = A(\lambda)$ and $x = B(\lambda)$ may not be nondecreasing everywhere, so that for some $x$ the inversion leads to a confidence region which is the union of two intervals, $[L_1(x), U_1(x)]$ and $[L_2(x), U_2(x)]$, where $U_1(x) < L_2(x)$. In such cases, the UACI may be redefined as the single interval $[L_1(x), U_2(x)]$, which has probability at least $1 - \alpha$ of containing $\lambda$. Examples of the UACI and this issue are shown in Section 4.

3. The tail functions approach

Under the model $(X \mid \lambda) \sim \text{Poisson}(\lambda)$, the tail functions approach can be used to define the $1 - \alpha$ generalized Garwood confidence interval (GGCI) for $\lambda$. This CI is $[L(x), U(x)]$, where $L(x)$ and $U(x)$ are the solutions for $\lambda$ in the two equations:

$$P(X \geq x \mid \lambda) = \alpha (1 - \tau(\lambda)),$$

$$P(X \leq x \mid \lambda) = \alpha \tau(\lambda)$$

respectively, except that $L(0) = 0$, where $\tau(\lambda)$ is a function whose domain is $(0, \infty)$ and whose range is a subset of $[0, 1]$. In this context we call $\tau(\lambda)$ the tail function. For simplicity we will only consider tail functions that are continuous and nondecreasing.

Theorem 2 in Appendix A shows that the $1 - \alpha$ GGCI contains $\lambda$ with a probability of at least $1 - \alpha$ for all possible values of $\lambda$. The ordinary Garwood confidence interval (OGCI) for $\lambda$, defined as per Garwood (1936), corresponds to the choice $\tau(\lambda) = \frac{1}{2}$, $\lambda \geq 0$, and is given by

$$\left[ \frac{X^2\alpha/2}{2}, \frac{X^2_{1-\alpha/2}(2(x + 1))}{2} \right],$$

where $X^2\alpha/2(a)$ is the (lower) $a$-quantile of the $\chi^2$ distribution with $a$ degrees of freedom. One-sided versions of the OGCI are defined by $\tau(\lambda) = 0$, $\lambda \geq 0$, and $\tau(\lambda) = 1$, $\lambda \geq 0$, respectively. For the former of these two versions, the upper bound of the CI is $\infty$ for all $x$, and for the latter the lower bound is 0 for all $x$.

The lines defining the bounds of the GGCI, namely $L = L(x)$ and $\lambda = U(x)$, may also be expressed in terms of the corresponding inverse functions $x = B(\lambda)$ and $x = A(\lambda)$ respectively, where for any specified value of $\lambda$, $B(\lambda)$ is the smallest integer $q$ such that

$$P(X > q \mid \lambda) \leq \alpha (1 - \tau(\lambda)).$$
and \( A(\lambda) \) is the largest integer \( r \) such that

\[
P(X < r \mid \lambda) \leq \alpha \tau(\lambda).
\]

(Note that functions \( A \) and \( B \) here are not the same as functions \( A \) and \( B \) in Section 2.) The following equation clarifies the inversions involved:

\[
1 - \alpha \leq P(A(\lambda) \leq X \leq B(\lambda) \mid \lambda)
= \sum_{x=A(\lambda)}^{B(\lambda)} f(x \mid \lambda)
= P(L(X) \leq \lambda \leq U(X) \mid \lambda)
= \sum_{x=0}^{\infty} f(x \mid \lambda) \mathbb{1}_{[L(x) \leq \lambda \leq U(x)]}.
\] (2)

If the Poisson mean is constrained in an interval \([a, b]\), a suitable tail function is one which equals 0 for \( \lambda \leq a \), equals 1 for \( \lambda \geq b \), and connects the points \((a, 0)\) and \((b, 1)\) continuously and nondecreasingly. Such a tail function may be thought of as a ‘graduated blend’ of the two one-sided versions of the OGCI as described above. It effectively ‘twists’ the OGCI bounds so that the lower bound asymptotes towards \( b \) as \( x \) increases, and the upper bound converges to a value no greater than \( a \) as \( x \) decreases towards zero. The result, after elimination of all impossible values (those less than \( a \) or greater than \( b \)), is a CI which is never empty and converges intuitively towards \( a \) or \( b \) as \( x \) decreases or increases respectively. The next section illustrates this behaviour.

4. Example with comparisons

Suppose that \( X \sim \text{Poisson}(\lambda) \), where \( \lambda \) has a prior distribution given by \( \lambda/10 \sim \text{Beta}(9, 2) \), so that \( \lambda \) is bounded above by 10, with prior mean \( 10 \times 9/(9 + 2) = 8.182 \) and prior mode \( 10 \times (9 - 1)/(9 + 2 - 2) = 8.889 \). Also suppose that we desire an 80% CI for \( \lambda \). That is, consider the case \( \alpha = 0.2 \), \( a = 0 \), and \( b = 10 \). Also suppose that we restrict our attention to the class of tail functions defined by the straight lines connecting \((0, 0)\), \((c, k)\), \((10, 1)\), and \((\infty, 1)\) in the \((\lambda, \tau(\lambda))\) plane. Here, \( c \in [0, 10) \) and \( k \in [0, 1) \) are tuning parameters that completely define \( \tau(\lambda) \), and our task is to search for the optimal values of these two parameters in some meaningful sense.

One criterion for optimality is prior expected length (PEL), which is defined as

\[
\text{PEL} = E[U(X) - L(X)] = \int_0^{\infty} \left\{ \sum_{x=0}^{\infty} [U(x) - L(x)] f(x \mid \lambda) \right\} f(\lambda) \, d\lambda.
\]

Here, \( f(x \mid \lambda) = e^{-\lambda} \lambda^x / x! \) as before, and \( f(\lambda) \) denotes the prior density of \( \lambda \). In the case of the UACI, the PEL is fixed, whereas for the GGCIs defined by the class of tail functions above, the PEL is a function of \( c \) and \( k \). Therefore a reasonable goal is to find the values of \( c \) and \( k \) that minimise the PEL.

Figure 1 shows three CIs for the case \( \alpha = 0.2 \), \( a = 0 \), and \( b = 10 \), these being the OGCI, the UACI and the GGCI defined by (1) with \( c = 0 \) and \( k = 0 \) (or, equivalently, \( c = 10 \) and...
Figure 1: Three 80% CIs: the ordinary Garwood, unified approach, and generalized Garwood. The OGCI is empty for $x > 14$, and the lower bound of the GGCI converges to 10 faster than that of the UACI. The OGCI is shown in both subfigures for comparative purposes.

Figure 2: The PELs of two CIs. The PEL of the UACI is 4.60 (straight horizontal dashed line), and that of the GGCI is shown as a function of $c$ for each $k = 0, 0.1, \ldots, 0.9$ (curved lines, from bottom to top). The bottom dot shows the minimum PEL, which occurs at $k = 0$ and $c = 7.3$. The other dot shows the minimum PEL when $k$ is fixed at 0.1; under that constraint the optimal value of $c$ is 7.1.

$k = 1)$. For example, if $x = 4$, these three CIs are $[1.74, 7.99]$, $[1.47, 7.47]$, and $[2.08, 7.31]$ respectively. It will be observed that after truncation the OGCI does not exist for $x > 14$, whereas the other two CIs exist for all $x = 0, 1, 2, \ldots$. Note that, by consideration of (2), the two lines defining the GGCI may be thought of in two ways: horizontally, according to $\lambda = L(x)$ and $\lambda = U(x)$, and vertically, according to $x = B(\lambda)$ and $x = A(\lambda)$ respectively.
A peculiar feature of the UACI is that sometimes it is actually a region made up of two disjoint intervals, for example $[8.43, 8.53]$ and $[8.90, 10]$ at $x = 14$. We can combine these two intervals and the space separating them so as to define a modified UACI, $[8.43, 10]$. If this modification is implemented generally, the result is a single interval that has coverage.
Figure 5: The frequentist coverage probabilities (FCPs) of four 80% CIs: (a) unified approach, (b) ordinary Garwood, (c) optimal generalized Garwood, and (d) choice generalized Garwood. Part (b) also shows the probability of the OGCI being nonempty (dashed line).

probability of at least 80% for all possible $\lambda$. We will note that no such modification is required for the GGCI.

Figure 2 shows 4.60, the PEL of the UACI (modified as above), and the PEL of the GGCI as a function of $c$ for various values of $k$ (0, 0.1, ..., 0.9). The minimum value of the PEL is 4.11 at $k = 0$ and $c = 7.3$, and this value is shown as a dot. Thus the ‘optimal’ GGCI, defined by these values of $c$ and $k$, is 10.6% shorter on average than the UACI. The other dot in Figure 2 (discussed further below) shows a PEL of 4.17 at $k = 0.1$ and $c = 7.1$ (the value of $c$ which minimises the PEL when $k$ is fixed at 0.1).

Prior expected length is not the only criterion by which the GGCI may be judged. Another, amongst many, is the maximum percentage by which the GGCI can possibly be wider than the UACI, a measure we will refer to as the maximum percentage wider (MPW). Figure 3 shows the MPW as a function of $c$ for each $k = 0, 0.1, \ldots, 0.9$. In particular, we find that the MPW is 302 at $(c, k) = (7.3, 0)$ and 141 at $(c, k) = (7.1, 0.1)$. These two MPWs are shown as dots in Figure 3 and correspond to the two dots in Figure 2. Thus, by changing the values of $c$ and $k$ slightly, and thereby incurring a slight increase in PEL, we can lower the MPW substantially. For this reason we may define the ‘choice’ GGCI in terms of the values $c = 7.1$ and $k = 0.1$.

Figure 4 shows the optimal and choice GGCIs, together with the UACI in Figure 1. It will be seen that at $x = 0$ and $x = 1$, the length of the choice GGCI is considerably less than the
length of the optimal GGCI. The lengths of these two CIs at \( x = 0 \) are 4.39 and 7.31, which are 141\% and 302\% wider than 1.82, the length of the UACI at \( x = 0 \). (From this we see that the two MPWs mentioned above occur at \( x = 0 \).)

To conclude this example we refer to Figure 5 which shows the frequentist coverage probabilities of the OGCI, the UACI, the optimal GGCI and the choice GGCI of Figures 1 and 4. It will be observed that each CI has a coverage of at least 80\% for all \( \lambda \). Note that the noncoverage of the OGCI, which is never more than 20\%, comprises two components, namely the probability of the CI existing and not containing \( \lambda \), and the probability of it being empty and – for that reason – not containing \( \lambda \). One of the lines in Figure 5 shows 80\% plus the first of these two probabilities, namely the overall probability of the OGCI existing (being nonempty). This probability slowly decreases from 1 at \( \lambda = 0 \) to 0.916 at \( \lambda = 10 \).

5. Summary and discussion

In this paper we have shown how to construct an exact CI for the Poisson mean \( \lambda \) when \( \lambda \) is bounded above, below, or both, and when a guarantee is required that the CI not be empty. The proposed method involves specifying a suitable class of tail functions \( \tau(\lambda) \), using each function in the class to define an exact CI for \( \lambda \), and then choosing the CI which is optimal according to some criterion or criteria. If prior information is available, one criterion is prior expected length. In this case the Bayesian approach could be used to provide a shorter interval on average but at the expense of it not having the required minimum frequentist coverage probability for all possible values of \( \lambda \). The ‘ordinary’ Garwood CI is defined by \( \tau(\lambda) = \frac{1}{2}, \lambda \geq 0 \), (constant for all \( \lambda \)), and leads to an empty CI with some probability that is a function of \( \lambda \).

The theory of tail functions was introduced in Puza and O’Neill (2005), (2006a), (2006b), and the present paper provides further details of this theory and its application, specifically to the problem of exact constrained interval estimation of the Poisson mean. This paper is closely related to Puza and O’Neill (2009) which dealt with constrained confidence estimation of the binomial proportion and a generalisation of the CI in Clopper and Pearson (1934), since the Poisson distribution is a limiting case of the binomial. This connection features in Appendix A, where the key result, Theorem 2, is proved using Theorem 1 as taken from Puza and O’Neill (2009). The present paper is also related to Puza and O’Neill (2008) which dealt with constrained interval estimation of the normal mean, since the Poisson distribution is approximately normal if \( \lambda \) is ‘large’.

As a final note we provide some numerical results in relation to the KARMEN experiment mentioned in Section 1. In this experiment there was an overall count of zero events, for which the 90\% OGCI for the mean of the overall count is \([0, 3.00]\). This then becomes \([2.88, 3.00]\) after elimination of all impossible values (those below the assumed background mean of 2.88). A direct application of the unified approach yields a 90\% confidence region consisting of the two intervals, \([2.88, 3.57]\) and \([3.59, 3.89]\), which may then be modified to produce the single 90\% UACI \([2.88, 3.89]\). The 90\% UACI reported in Mandelkern (2002) is \([2.88, 3.95]\) (or \([0, 1.07]\) for the signal mean), but this result involves a further modification of 3.89 to 3.95, the details of which are unclear to the present authors after studying the relevant papers cited in Mandelkern (2002). By contrast, using a simple tail function which linearly connects the points \((0, 0), (2.88, 0), (3.5, 0.5), (\infty, 0.5)\) in the \((\lambda, \tau(\lambda))\) plane, we obtain the 90\% GGCI \([2.88, 3.33]\), which is shorter and requires no modification.

In this example we might argue that there is no need for anything but the OGCI, since the smallest possible count (zero) yields a 90\% OGCI that is not entirely below the lower bound (2.88). However, the 80\% OGCI for the overall mean is \([0, 2.30]\), which does become
empty after truncation. Also, a direct application of the unified argument yields the 80% CI [2.88, 3.27], and the tail function defined in the previous paragraph results in the 80% GGCI [2.88, 3.15]. Thus, for purposes of constructing an 80% (rather than 90%) CI, the ordinary Garwood approach fails, and the tail functions approach results in a narrower interval than the unified approach.

Appendix A.

Theorem 1. Suppose that \( Y_n \sim \text{Bin}(n, p) \), where \( n \in \{1, 2, 3, \ldots \} \) and \( 0 \leq p \leq 1 \), and let \( \tau_n(p) \) be a nondecreasing function whose domain is \([0, 1]\) and whose range is a subset of \([0, 1]\). Then, for a given \( \alpha \in (0, 1) \), define the functions \( L_n \) and \( U_n \) such that

\[
L_n(y) \text{ is the solution for } p \text{ in } P(Y_n \geq y \mid p) = \alpha(1 - \tau_n(p)), \quad y = 1, \ldots, n,
\]

\[
U_n(y) \text{ is the solution for } p \text{ in } P(Y_n \leq y \mid p) = \alpha \tau_n(p), \quad y = 0, \ldots, n - 1,
\]

\[
L_n(0) = 0,
\]

\[
U_n(n) = 1.
\]

Then \( L_n(y) \) and \( U_n(y) \) are unique, and \( P(L_n(Y_n) \leq p \leq U_n(Y_n)) \geq 1 - \alpha \).

Proof. See Puza and O’Neill (2009), where the notation used is slightly different.

Theorem 2. Suppose that \( X \sim \text{Poisson}(\lambda) \), where \( \lambda \geq 0 \), and let \( \tau(\lambda) \) be a nondecreasing function whose domain is \([0, \infty)\) and whose range is a subset of \([0, 1]\). Then, for a given \( \alpha \in (0, 1) \), define the functions \( L \) and \( U \) such that

\[
L(x) \text{ is the solution for } \lambda \text{ in } P(X \geq x \mid \lambda) = \alpha(1 - \tau(\lambda)), \quad x = 1, 2, 3, \ldots,
\]

\[
U(x) \text{ is the solution for } \lambda \text{ in } P(X \leq x \mid \lambda) = \alpha \tau(\lambda), \quad x = 0, 1, 2, \ldots,
\]

\[
L(0) = 0.
\]

Then \( L(x) \) and \( U(x) \) are unique, and \( P(L(X) \leq \lambda \leq U(X)) \geq 1 - \alpha \).

Proof. Theorem 2 can be proved using the theory of randomized CIs in much the same way as Theorem 1 is proved in Puza and O’Neill (2009).

Another proof is to take Theorem 1 as a starting point and apply the definition of the Poisson distribution as a limiting case of the binomial. Adopting this second approach, for any integer \( n \) let \( p = \lambda / n \) and define \( \tau_n(p) = \tau(np) \) for \( 0 \leq p \leq 1 \). Then \( \tau_n(p) \) is a nondecreasing function whose domain is \([0, 1]\) and whose range is a subset of \([0, 1]\). So by Theorem 1 there exist unique functions \( L_n \) and \( U_n \) such that

\[
L_n(y) \text{ is the solution for } \frac{\lambda}{n} \text{ in } P\left(Y_n \geq y \mid p = \frac{\lambda}{n}\right) = \alpha \left(1 - \tau_n\left(\frac{\lambda}{n}\right)\right), \quad y = 1, \ldots, n,
\]

\[
U_n(y) \text{ is the solution for } \frac{\lambda}{n} \text{ in } P\left(Y_n \leq y \mid p = \frac{\lambda}{n}\right) = \alpha \tau_n\left(\frac{\lambda}{n}\right), \quad y = 0, \ldots, n - 1,
\]

\[
L_n(0) = 0,
\]

\[
U_n(n) = 1,
\]

\[
P\left(L_n(Y_n) \leq \frac{\lambda}{n} \leq U_n(Y_n)\right) \geq 1 - \alpha.
\]
Or equivalently, writing $P(Y \geq y \mid p = \lambda/n)$ as $P(Y \geq y \mid \lambda)$, (3) can be stated as follows:

$$n \ln(y)$$ is the solution for $\lambda$ in $P(Y \geq y \mid \lambda) = \alpha(1 - \tau(\lambda)), \quad y = 1, \ldots, n,$

$$nU_n(y)$$ is the solution for $\lambda$ in $P(Y \leq y \mid \lambda) = \alpha \tau(\lambda), \quad y = 0, \ldots, n - 1,$

$$nL_n(0) = 0,$$

$$nU_n(n) = n,$$

$$P(nL_n(Y_n) \leq \lambda \leq nU_n(Y_n)) \geq 1 - \alpha.$$

Now, with $np$ fixed at $\lambda$, $Y_n \sim \text{Bin}(n, p)$ converges in distribution to $X \sim \text{Poisson}(\lambda)$ as $n \to \infty$. Therefore, in that limit there exist unique functions $L$ and $U$, given by $L(x) = \lim_{n \to \infty}(nL_n(x))$ and $U(x) = \lim_{n \to \infty}(nU_n(x))$ (possibly infinity), such that

$$L(x)$$ is the solution for $\lambda$ in $P(X \geq x \mid \lambda) = \alpha(1 - \tau(\lambda)), \quad x = 1, 2, 3, \ldots,$

$$U(x)$$ is the solution for $\lambda$ in $P(X \leq x \mid \lambda) = \alpha \tau(\lambda), \quad x = 0, 1, 2, \ldots,$

$$L(0) = 0,$$

$$P(L(X) \leq \lambda \leq U(X)) \geq 1 - \alpha.$$

References


