A BLACK–SCHOLES MODEL WITH GARCH VOLATILITY

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Abstract

Option pricing based on the Black–Scholes model is typically obtained under the assumption that the volatility of the return is a constant. In this paper, we develop a new method for pricing derivatives under the Black–Scholes model with GARCH volatility by viewing the call price as an expected value of a truncated normal distribution. Using return data, we estimate the mean, variance, and kurtosis of the random volatility in the Black–Scholes model. An extensive empirical analysis of the European call option valuation of the S&P 100 Index shows that our method outperforms other competing GARCH pricing models and the pricing errors when using our method are quite small even though our estimation procedure is based only on the historical return data.

Keywords: GARCH volatility; Black–Scholes formula; GARCH option pricing; Monte Carlo

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Secondary 82B31

1. Introduction

Option pricing is one of the major areas in modern financial theory and practice. Since Fischer Black, Myron Scholes, and Robert Merton introduced their path-breaking work on option pricing, there has been an explosive growth in derivatives trading activities in the worldwide financial markets. The main contribution of the seminal work of Black and Scholes (1973) and Merton (1973) was the introduction of an option pricing formula that does not involve an investor’s risk preferences and subjective views. The compact form and computational simplicity of the Black–Scholes formula ensures popularity in the finance industries. There is a general consensus that asset returns exhibit variances that change through time (see, for example, Taylor (2007)), and GARCH coefficient Black–Scholes models are a useful choice to model those changing variances. Option pricing under a GARCH volatility is thus an interesting and practically relevant topic in modern financial analysis.

Duan (1995) was the first to provide a solid theoretical foundation based on the concept of a locally risk-neutral option valuation relationship for option valuation under GARCH models using Monte Carlo simulation. Recently Heston and Nandi (2000), Elliot et al. (2006), Badescu and Kulperger (2008), Barone-Adesi et al. (2008), and Mercuri (2008), among others, have derived closed form option pricing formulas under various GARCH models for volatility.
this paper, we propose a new approach to compute the call price based on a conditional Black–Scholes model with GARCH volatility, by viewing the call price as a first conditional moment of a truncated lognormal distribution under the martingale measure.

This paper is organized as follows. In Section 2 we present the conditional Black–Scholes model with volatility modeled by a GARCH process. Based on the estimates of the moments of the GARCH process, a modified option pricing formula is given. The results of an empirical study based on the European call option valuation of S&P 100 data are also given.

2. Option pricing with GARCH volatility

We assume that the price process $S_t$ follows the conditional Black–Scholes model since, given $\theta_t = \sigma$,

$$dS_t = rS_t \, dt + \theta_t S_t \, dW_t \quad \text{and} \quad y_t = \log \frac{S_t}{S_{t-1}} - \mathbb{E} \left[ \log \frac{S_t}{S_{t-1}} \right] = \theta_t Z_t,$$

where $r$ is the risk-free interest rate, $\{W_t\}$ is a standard Brownian motion and $\{\theta_t\}$ is the volatility process. Then, conditional on $\theta_t$, the call price is given by

$$C(S, T) = e^{-rT} \mathbb{E} \left[ \{S_T - K\}^+ \right]$$

$$= e^{-rT} \mathbb{E}_0 [\mathbb{E} \left[ \{S_T - K\}^+ \right]]$$

$$= S \mathbb{E}_0 [f(\theta_t)] - K e^{-rT} \mathbb{E}_0 [g(\theta_t)],$$

where

$$f(\theta_t) = \Phi \left( \frac{\log(S/K) + rT + \frac{1}{2} \theta_t^2}{\theta_t} \right), \quad g(\theta_t) = \Phi \left( \frac{\log(S/K) + rT + \frac{1}{2} \theta_t^2}{\theta_t} - \theta_t \right),$$

$\{S_T - K\}^+ = \max[\{S_T - K\}, 0]$, $S$ is the initial value of $S_t$, $K$ the strike price, and $T$ the expiry date. Also, $\theta_t$ is a stationary process having mean $\mu_\theta$, variance $\sigma_\theta^2$, skewness $\gamma_\theta$, and kurtosis $\kappa(\theta)$.

**Theorem 1.** For any twice continuously differentiable functions $f(x)$ and $g(x)$, the call price is

$$C(S, T) = S \mathbb{E}_0 [f(\theta_t)] - K e^{-rT} \mathbb{E}_0 [g(\theta_t)]$$

$$= S \mathbb{E}_0 \left[ f(\mathbb{E}(\theta^2_t)) \right] + \frac{1}{2} f''(\mathbb{E}(\theta^2_t)) \mathbb{E}_0 \left[ (\mathbb{E}(\theta^2_t) + 2\mathbb{E}^2(\theta_t)) (\kappa(\theta) + 4 \mathbb{E}^2(\theta_t) - 1) \right]$$

$$- K e^{-rT} \left[ g(\mathbb{E}(\theta^2_t)) \mathbb{E}_0 \left[ (\mathbb{E}(\theta^2_t) + 2\mathbb{E}^2(\theta_t)) (\kappa(\theta) + 4 \mathbb{E}^2(\theta_t) - 1) \right] \right]$$

$$= S \mathbb{E}_0 \left[ f(\mathbb{E}(\theta^2_t)) \right] + \frac{1}{2} f''(\mathbb{E}(\theta^2_t)) \left( \frac{1}{3} \kappa(\theta^2) - 1 \right) \mathbb{E}^2(\theta^2_t)$$

$$- K e^{-rT} \left[ g(\mathbb{E}(\theta^2_t)) \mathbb{E}_0 \left[ (\mathbb{E}(\theta^2_t) + 2\mathbb{E}^2(\theta_t)) \left( \frac{1}{3} \kappa(\theta^2) - 1 \right) \mathbb{E}^2(\theta^2_t) \right] \right].$$

where

$$\kappa(\theta) = \frac{\mathbb{E}[\theta_t - \mathbb{E}(\theta_t)]^4}{\mathbb{E}^2[\theta_t - \mathbb{E}(\theta_t)]^2}$$

is the kurtosis of volatility process $\theta_t$, and

$$\kappa(\theta^2) = \frac{\mathbb{E}(\theta_t^4)}{\mathbb{E}^2(\theta_t^2)}$$
A Black–Scholes model with GARCH volatility

the kurtosis of observed log-return \( y_t \). We also have

\[
f[E(\theta_t^2)] = \Phi(d) \\
= \Phi\left( \frac{\log(S/K) + rT + \frac{1}{2} E(\theta_t^2)}{\sqrt{E(\theta_t^2)}} \right),
\]

and

\[
g[E(\theta_t^2)] = \Phi(d - \sqrt{E(\theta_t^2)}) \\
= \Phi\left( \frac{\log(S/K) + rT - \frac{1}{2} E(\theta_t^2)}{\sqrt{E(\theta_t^2)}} \right)
\]

\[f''[E(\theta_t^2)] = \frac{1}{\sqrt{2\pi}} \left[ -\frac{(E(\theta_t^2) - 2(\log(S/K) + rT))}{4 E(\theta_t^2)^2} \left( \frac{[E(\theta_t^2)]^2 - 4(\log(S/K) + rT)^2}{8[E(\theta_t^2)]^2} \right) \right. \\
+ \left. \frac{6(\log(S/K) + rT) - E(\theta_t^2)}{8[E(\theta_t^2)]^2} \right] \]
\[\times \exp\left[ -\frac{(2(\log(S/K) + rT) + E(\theta_t^2))^2}{8 E(\theta_t^2)^2} \right].
\]

\[g''[E(\theta_t^2)] = \frac{1}{\sqrt{2\pi}} \left[ \frac{(E(\theta_t^2) + 2(\log(S/K) + rT))}{4 E(\theta_t^2)^2} \left( \frac{[E(\theta_t^2)]^2 - 4(\log(S/K) + rT)^2}{8[E(\theta_t^2)]^2} \right) \right. \\
+ \left. \frac{6(\log(S/K) + rT) + E(\theta_t^2)}{8[E(\theta_t^2)]^2} \right] \]
\[\times \exp\left[ -\frac{(2(\log(S/K) + rT) - E(\theta_t^2))^2}{8 E(\theta_t^2)^2} \right].
\]

**Proof.** The derivatives of \( f \) and \( g \) are

\[
f'(\theta_t) = \frac{1}{\sqrt{2\pi}} \left[ \frac{\theta_t - 2(\log(S/K) + rT)}{4\theta_t} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) + \theta_t)^2}{8\theta_t} \right].
\]

\[
g'(\theta_t) = \frac{1}{\sqrt{2\pi}} \left[ \frac{-\theta_t - 2(\log(S/K) + rT)}{4\theta_t} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) - \theta_t)^2}{8\theta_t} \right].
\]

\[
f''(\theta_t) = \frac{1}{\sqrt{2\pi}} \left[ \frac{\theta_t - 2(\log(S/K) + rT)}{4\theta_t \sqrt{\theta_t}} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) + \theta_t^2)}{8\theta_t^2} \right] \\
+ \left[ \frac{6(\log(S/K) + rT) - \theta_t^2}{8\theta_t^2 \sqrt{\theta_t}} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) + \theta_t^2)^2}{8\theta_t^2} \right].
\]

\[
g''(\theta_t) = \frac{1}{\sqrt{2\pi}} \left[ \frac{\theta_t + 2(\log(S/K) + rT)}{4\theta_t \sqrt{\theta_t}} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) - \theta_t)^2}{8\theta_t^2} \right] \\
+ \left[ \frac{6(\log(S/K) + rT) + \theta_t}{8\theta_t^2 \sqrt{\theta_t}} \right] \exp\left[ -\frac{(2(\log(S/K) + rT) - \theta_t)^2}{8\theta_t^2} \right].
\]

and the proof follows.
Example 1. (GARCH\((p, q)\) model for volatility.) Option pricing based on GARCH models has been studied under the assumption that the innovations are standard normal (i.e., under normal GARCH), that is
\[
dS_t = rS_t \, dt + \theta_t S_t \, dW_t, \\
y_t = \log \frac{S_t}{S_{t-1}} - \mathbb{E} \left[ \log \frac{S_t}{S_{t-1}} \right] = \theta_t Z_t, \\
Z_t \overset{i.i.d}{\sim} \mathcal{N}(0, 1),
\]
where
\[
\theta_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \theta_{t-j}^2.
\]
However, the usual option pricing formula based on GARCH models does not depend on the skewness and the leptokurtosis of the observed process. In Theorem 1, the estimates of \(E(\theta_t^2)\) and \(\kappa^{(\gamma)}\) can be obtained by using
\[
E(\theta_t^2) = \frac{\omega}{1 - \sum_{j=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j} \quad \text{and} \quad \kappa^{(\gamma)} = \frac{3}{1 - 2 \sum_{j=1}^{\infty} \psi_j^2},
\]
where the \(\psi_j\)s are obtained from the relation \(\psi(B) \phi(B) = \beta(B)\) with \(\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i\), \(\phi(B) = 1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j\), \(\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j\), and \(B\) is the lag (back) operator which is defined as \(y_{t-1} = B y_t\).

Moreover, for a Black–Scholes model with GARCH\((1, 1)\) volatility, that is
\[
dS_t = rS_t \, dt + \theta_t S_t \, dW_t, \\
y_t = \log \frac{S_t}{S_{t-1}} - \mathbb{E} \left[ \log \frac{S_t}{S_{t-1}} \right] = \theta_t Z_t, \\
\theta_t^2 = \omega + \alpha y_{t-1}^2 + \beta \theta_{t-1}^2,
\]
the moments \(E(\theta_t^2)\) and \(\kappa^{(\gamma)}\) are given by
\[
E(\theta_t^2) = \frac{\omega}{1 - \alpha - \beta} \quad \text{and} \quad \kappa^{(\gamma)} = \frac{3}{1 - 2 \alpha^2 / (1 - (\alpha + \beta)^2)}
\]
respectively.

2.1. Empirical results

In this section, we present numerical results for the comparison of our proposed method with the standard Black–Scholes formula and Duan’s (1995) Monte Carlo method (see Taylor (2005) for details), using S&P 100 daily index series from 2 January 1991 to 29 December 2000 with 2530 observations. The data were obtained from http://pup.princeton.edu/titles/8055.html.

The option prices are obtained using several methods. Here we compare the call prices based on our proposed method with the call prices calculated from the Black–Scholes formula and Duan’s (1995) Monte Carlo Simulation based on GARCH methods.

The observed call prices are compared with the estimated prices obtained from the models mentioned above. For the short, medium, and long term expiry dates, the call prices estimated by our proposed method are closer to the observed call prices than those of the other models. The percentage error in Tables 1 and 2 is calculated in the following way:
\[
\% \, \text{error} = \frac{|\text{observed call price} - \text{estimated call price}|}{\text{observed call price}} \times 100\%.
\]
3. Conclusions

In this paper, a partially Bayes approach for option pricing has been proposed. In the Black–Scholes model the volatility is assumed to follow a GARCH process as in Thavaneswaran et al. (2006) and, by conditioning on the volatility, the call price is first expressed as an expectation (with respect to random volatility) and then evaluated in terms of the moments of the volatility process. A numerical example demonstrates the superiority of the approach.

References


Table 2: Comparison of Black–Scholes call prices, call prices obtained by the GARCH model Monte Carlo simulation and the proposed method.

<table>
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<tr>
<th>$T$</th>
<th>$S$</th>
<th>$K$</th>
<th>Observed Call price</th>
<th>Black–Scholes</th>
<th>% Error</th>
<th>Monte Carlo</th>
<th>% Error</th>
<th>GARCH</th>
<th>% Error</th>
<th>Proposed</th>
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