A SIMPLE PARRONDO PARADOX

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Abstract

The Parrondo paradox is a counterintuitive probabilistic phenomenon in situations of repeated decision making with uncertainty. It can be simply stated as follows: given two games, each with a higher probability of losing than winning, it is possible to construct a winning strategy by playing the games alternately. The best-known and analyzed examples are somewhat involved, and may not be relevant to practical situations of repeated decision making, e.g. casino games and investment in securities markets. Here, we construct a very simply stated and analyzed example of the Parrondo paradox, using a model of stylized Blackjack-type betting that is well-known in the gambling literature.

Keywords: Parrondo paradox; Kelly criterion

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1. Introduction

The online edition of the Cambridge Dictionary (see http://dictionary.cambridge.org) defines a paradox to be ‘a situation or statement which seems impossible or is difficult to understand because it contains two opposite facts or characteristics’. Some well-known paradoxes are statistical phenomena. For example, Lindley’s paradox of statistical inference occurs when a classical test of the null hypothesis strongly rejects it, while a Bayesian posterior odds ratio highly favors it [10]. Of course, the phenomenon can be understood; it is just difficult to do so without further analysis. Another example is Parrondo’s paradox, which can be posed as follows (see [5]):

‘But can two losing gambling games be set up such that, when they are played one after the other, they becoming winning? The answer is yes’.

Key et al. [7] demonstrated this in situations where play switches from one simple game to another whenever the total fortune is divisible by an integer determined in advance. They generalized this to incorporate alternative switching rules, including the possibility of switching plays on the result of a coin toss. Di Crescenzo [2] developed a Parrondo paradox in the reliability of chained components, by exploiting the fact that the product of mixtures of survival functions stochastically dominates the product of the component survival functions. Berresford and Rockett [1] contains a nice exposition of the earlier Parrondo paradox examples, while Ethier and Lee [4] devised a framework for generalizing and analyzing the earlier examples. Here, we demonstrate that a Parrondo paradox may arise in a simple model of Blackjack-type betting, which has been discussed by Ed Thorp [11], a famous inventor of card-counting Blackjack strategies. The required switching rules are also in accord with reality – they are the two most common ways of implementing the oft-advocated principle of diversification.

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2. Blackjack-type betting

The popular stylized model of Blackjack betting, discussed by Thorp [11, Chapter 9], MacLean et al. [8, pp. 1575–1577], and others is now described. The bettor chooses a constant fraction $f$ of their table fortune to bet on each (Bernoulli) play, having a gross return per dollar invested equal to $1 + f$ with probability $\pi > \frac{1}{2}$, or $1 - f$ with probability $1 - \pi$. It is assumed that the bet is favorable, i.e. $\pi > \frac{1}{2}$, so the bet has a positive ‘edge’ $(2\pi - 1)f$ per play. Hence, by what is popularly referred to as ‘Arrow’s theorem’, any risk-averse (i.e. expected strictly concave utility of return) bettor will want to bet some fraction of fortune $f > 0$.

For example, suppose that a person starts with $F_0 = $10,000 and chooses to bet $f = 5\%$ on each play. So they bet $10,000 \times 0.05 = $500 on the first play. If they win, they will have $10,000 \times 1.05 = 10,500$ available for the next play. They then bet $10,500 \times 0.05 = $525. If they win again, they will have $10,500 \times 1.05 = $11,025, but if they lose, they will have $10,500 \times 0.95 = $9975. These Bernoulli trials continue.

MacLean et al. [8, p. 1576] reported the well-known finding that successful ‘card counting’ Blackjack strategy can result in a favorable edge, see e.g. Thorp [11], and that ‘an approximation that will provide us insight into the long-run behavior of a player’s fortune is to simply assume that the game is a Bernoulli trial with a probability of success $\pi = 0.51$ and probability of loss $1 - \pi = 0.49$’.

The bettor’s fortune after $n$ plays, denoted by $F_n$, will be

\[
F_n = F_0[(1 + f)^w (1 - f)^{n-w}] = F_0 \exp(\log((1 + f)^w (1 - f)^{n-w})) = F_0 \exp(w \log(1 + f) + (n - w) \log(1 - f)) = F_0 \exp\left(\frac{w}{n} \log(1 + f) + \frac{n - w}{n} \log(1 - f)\right) n, \tag{1}
\]

where $w$ is the binomially distributed number of winning bets. The bettor’s fortune $F_n$ is the product of $n$ independent and identically distributed (i.i.d.) random variables, so its expected value after $n$ plays is the compounded value of its expected value per play, i.e.

\[
E[F_n] = F_0[\pi(1 + f) + (1 - \pi)(1 - f)]^n. \tag{2}
\]

With $\pi = 0.51$ and $f = 5\%$, $E[F_n] = 1.001^n F_0$, so after, say, $n = 1000$ repeated plays, the expected value of the fortune is $2.717 F_0$. But this statistic is extremely misleading. The fortune $F_{1000}$ is extremely positively skewed. To see this, recall that the binomially distributed number of wins $w$ is approximately normal when $\pi$ is close to 0.5 and $n$ is suitably large. Equation (1) then shows that $F_n$ is approximately log-normal, and thus positively skewed. As such, its expected value is atypically high, much higher than the lower median. In fact, Ethier [3] showed that for suitably large $n$ in many i.i.d. processes (including this one), a reasonable approximation to the median wealth is made by using the expected log gross return per play as an asymptotic exponential growth factor for the median. The approximation is

\[
\text{median}[F_n] \approx F_0 \exp[\pi \log(1 + f) + (1 - \pi) \log(1 - f)] n) = F_0 \exp(-0.00025 \times 1000) = 0.778 F_0. \tag{3}
\]
A simple Parrondo paradox

Figure 1: Histogram from 234 simulations of $F_{1000}$, starting with $F_0 = 10000$, showing that the fortune is positively skewed.

Table 1.

<table>
<thead>
<tr>
<th>$F_{1000}$</th>
<th>Frequency</th>
<th>Cumulative percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 500</td>
<td>55</td>
<td>24%</td>
</tr>
<tr>
<td>5 000</td>
<td>31</td>
<td>37%</td>
</tr>
<tr>
<td>7 780</td>
<td>33</td>
<td>51%</td>
</tr>
<tr>
<td>10 000</td>
<td>11</td>
<td>56%</td>
</tr>
<tr>
<td>12 500</td>
<td>10</td>
<td>60%</td>
</tr>
<tr>
<td>15 000</td>
<td>14</td>
<td>66%</td>
</tr>
<tr>
<td>17 500</td>
<td>13</td>
<td>71%</td>
</tr>
<tr>
<td>20 000</td>
<td>5</td>
<td>74%</td>
</tr>
<tr>
<td>22 500</td>
<td>5</td>
<td>76%</td>
</tr>
<tr>
<td>25 000</td>
<td>4</td>
<td>77%</td>
</tr>
<tr>
<td>27 170</td>
<td>5</td>
<td>79%</td>
</tr>
<tr>
<td>More</td>
<td>48</td>
<td>100%</td>
</tr>
</tbody>
</table>

that is, the approximate median outcome is a loss of around 22% of the initial stake. Because the median outcome is a loss of fortune, there is a higher probability of losing income in the repeated game than in winning income. This is quite disturbing, and illustrates both the risk of overly aggressive betting in a favorable game and the spectacular failure of the expected value (a gain of 172%) as a measure of central tendency in a highly skewed distribution of outcomes.

An alternative confirmation of this is enabled by simulating values of $F_n$ on a computer spreadsheet. Its random number generator may be used to simulate the 0.51 biased coin toss. Each time a ‘head’ is tossed, multiply $F_{n-1}$ by 1.05. If a ‘tail’ is tossed, multiply $F_{n-1}$ by 0.95. Starting with $F_0 = 10000$ and doing this for 1000 simulated tosses results in a simulated value for $F_{1000}$. Figure 1 depicts a histogram from 234 simulated values of $F_{1000}$, illustrating the positive skewness, and showing that the resulting simulated median is far lower than the population expected value near 27 120 (see Table 1 for the data). Of course, there is simulation error in the simulated value for the median, but the cumulated histogram shows that the simulated median is close to the value calculated in (3).

Even without the results of this simulation, the validity of the approximation (3) can be determined. We note that the median number of wins is (the closest integer to) $n\pi = 510$,
and that (1) is a monotone increasing transformation \( g \) of the number of wins \( w \). Because
\[
\text{median}[g(w)] = g(\text{median}[w])
\]
for any \( g \) that is monotone increasing, (1) implies that
\[
\text{median}[F_n] = F_0 \exp\left(\frac{n\pi}{n} \log(1 + f) + \frac{n - n\pi}{n} \log(1 - f)\right)
\]
\[
= F_0 \exp(-0.00025 \times 1000)
\]
\[
= 0.778 F_0.
\]
showing that the approximation (3) for i.i.d. plays is exact in our biased coin-toss special case.

In other i.i.d. cases, Ethier [3] showed that an even more accurate approximation incorporates the degree of skewness in the log gross return per play, but this refinement is not needed.

3. Diversification mechanisms producing a Parrondo paradox

Now suppose that the bettor plays two identical games alternately. He diversifies, placing half his funds, i.e. $5000, in each game. He then alternates play between the two, and lets the funds ‘ride’ in each game. His total fortune after \( n \) plays of both games (determined by \( 2n \) Bernoulli trials) is
\[
F_n(\text{diversified ride}) = \frac{1}{2} F_0[(1 + f)^{w_1} (1 - f)^{w_1-n}] + \frac{1}{2} F_0[(1 + f)^{w_2} (1 - f)^{w_2-n}]. \tag{4}
\]
Because \( \pi = 0.51 \) is close to one half, the binomially distributed number of wins is approximately normally distributed, and using (1), we see that the sum of two independent log-normal distributions may be used to approximate the distribution of (4). Unfortunately, that distribution does not have a closed form. So the aforementioned spreadsheet simulation was modified to simulate values of (4). Divide \( F_0 = 10000 \) in half and perform the simulation as above, to produce a simulated value of the first term in (4). Repeat this process to produce a simulated value of the second term of (4), and add the two results to produce a simulated value of (4).
To reduce simulation error, this process is repeated 10 000 times and the simulated median is found as
\[
\text{median}[F_n(\text{diversified ride})] \approx 1.2 F_0.
\]
Summarizing, we see that repeated play of the single game resulted in a median loss of around 22% of the initial stake, while merely splitting the initial stake between two such identical games resulted in a median gain of around 20%! The outcome after repeated play of each game is a higher probability of losing money than winning it. But splitting the initial stake between two such games, and playing them alternately in the same way that the single game was played, (that is, betting \( f = 5\% \) of the income at risk in each game) results in a higher probability of winning income than losing it. This is a Parrondo paradox.

The probabilistic mechanism underlying this result is simple. When only one game is played, the possible outcomes of the first play are just a gain of 5% with 0.51 probability or a loss of 5% with 0.49 probability. But when the money is split between the two games, there are three possible outcomes of the first play of both: a gain of 5% from \((5000 \times 1.05 + 5000 \times 1.05)\) with 0.51\(^2 = 0.2601\) probability, a loss of 0.05 with probability 0.49\(^2 = 0.2401\), and breaking even (from \(5000 \times 1.05 + 5000 \times 0.95 = 10000\)) with the complementary probability 0.4998. The nearly even odds of breaking even lowers the volatility of the first play, while leaving the expected value of the first play unchanged. This mechanism works to raise the median of (4).

It is, however, possible to do even better. Suppose that the bettor splits their initial stake equally between two identical and independently run Blackjack games, but now commits to
maintaining the 50–50 split of funds over the course of play. That is, they again start with \( F_0 = \$10,000 \), putting \$5000 to work in one game and the other \$5000 to work in the other game. They then execute one play of the first game, and then one play of the second game. If they win in both games, they will have \( 5000 \times 1.05 + 5000 \times 1.05 = 10,500 \). They will allocate half (\$5250) to each game before alternating play again. That is, the bettor lets the winnings ride in both games, as before. If instead they lost in both games, they would have \( 2 \times 5000 \times 0.95 = 9500 \), and would thus allocate \$4750 to each game before alternating play again (that is, let the losings ride in both games, as before). But if instead they win in one game while losing in the other, they would have \( 5000 \times 1.05 + 5000 \times 0.95 = 5250 + 4750 = 10,000 \), and would need to reallocate 250 from the winning game to the losing game, in order to maintain the 50% allocation weight in each game. This need to reallocate funds from the better-performing game into the worse-performing game will continue throughout play, whenever they win in one while losing in the other.

The expected value of the fortune \( F_n \) after \( n \) plays of this two-game rebalanced strategy is

\[
E[F_{n}(\text{rebalanced diversification})] = F_0[\pi^2 (1 + f) + (1 - \pi)^2 (1 - f) + 2\pi (1 - \pi) (1)]^n = F_0(1.001)^{1000} = 2.717F_0,
\]

which is exactly the same large expected gain as in the single (losing) game (2). But the median value will be radically different. To see this, compute the expected log utility of gross return per play to make the approximation

\[
\text{median}[F_n] \approx F_0 \exp[\pi^2 \log(1 + f) + (1 - \pi)^2 \log(1 - f) + 2\pi (1 - \pi) \log(1)] = F_0 \exp(0.0003748 \times 1000) = 1.45F_0.
\]

So rather than suffering a median loss of around 22% after 1000 plays of a single game, the alternating play of two identical games results in a median gain of approximately 45%, as long as the bettor continuously keeps their fortune split equally between the two games. This is substantially better than the 20% median gain achieved from the diversified ride.

### 4. Discussion

We might question whether a gambler could implement strategies like these in actual casino games. But we should not doubt that an investor could. The single game is equivalent to the binomial tree model of stock price evolution, which is used in standard undergraduate finance textbooks (see e.g. [6, Chapter 11]). Specifically, it is a model of a stock that either goes up or down by the same percentage each period with complementary probabilities (0.51 versus 0.49), and has a positive expected net return per period (here, 10 basis points). The diversified ride strategy is equivalent to buying and holding an (initially) equally value-weighted portfolio of two uncorrelated (but otherwise identical) stocks. The superior rebalanced diversification strategy is an equally value-weighted, continuously rebalanced portfolio of the two. This provides a spectacular demonstration of the value that both buy-and-hold and continuously rebalanced diversification strategies can have for investors.

It is interesting to examine the special case of someone who wants to maximize the long-run median gain. Using the Ethier [3] approximation, we will want to find the value of \( f \) that maximizes the expected log utility of return. This is also dubbed the ‘log-optimal’ or
‘Kelly’ criterion (see MacLean et al. [8]). In the single game with outcome (1), this will be
\[
\arg \max_f \pi \log(1 + f) + (1 - \pi) \log(1 - f) = 2\pi - 1.
\]
This is \(f = 2\%\) rather than \(f = 5\%\), and will result in a positive expected log gross return of 0.0002. As such, the single game will have a median gain, rather than the loss associated with \(f = 5\%\). It is easy to show that the log-optimal bet in a favorable game (i.e. \(\pi > \frac{1}{2}\)) will always achieve a positive expected log gross return and hence have a median gain. So that bettor cannot be the source of a Parrondo paradox.

Finally, while we could consider policies with time-varying values of \(f\), it is not hard to show that in i.i.d. environments, a bettor wanting to maximize expected log utility will not gain anything by doing so. Hence, when using the Kelly criterion, there is no loss in generality in assuming that \(f\) is fixed across time.

5. Conclusion

The Parrondo paradox that two losing games can be played in a way to produce a winning outcome is not just a probabilistic curiosity. A simplified model of Blackjack, used for analytic purposes by Ed Thorp (a famous inventor of a card-counting Blackjack technique) and other analysts, was used to produce a Parrondo paradox. Moreover, our demonstration of this used very elementary methods, including simulations that can be quickly performed by those with only basic expertise in the use of computer spreadsheets, in contrast to the more complex analyses of other Parrondo paradoxes that abound in the literature. Those wishing to extend the conceptual framework used here should examine Maslov and Zhang [9], who studied investment in an asset whose median price drifted downward over time. They constructed a rebalanced portfolio of that asset and cash, with median portfolio value drifting upward over time.

References