Dear Editor,

The transient probabilities of a simple immigration–emigration–catastrophe process

Consider a simple immigration–emigration process influenced by random total catastrophes which, when they occur, annihilate the entire population. Assume that the immigration rate \( \lambda > 0 \), the emigration rate \( \mu > 0 \) and the catastrophe rate \( \gamma > 0 \) do not depend on the population size \( \{N(t); t \geq 0\} \); the process can then be described as having the transition rates in the time interval \((t, t + \delta t)\) given in Table 1.

Krishna Kumar and Arivudainambi (2000) used the forward Kolmogorov equations to determine the probability \( p_n(t) \) that the population size \( N(t) \) is \( n \), given that \( P(N(0) = k) = p_k \), \( k \geq 0 \). After some manipulations, they obtained the Laplace transforms of \( p_n(t) \), \( n \geq 0 \), which on inversion yielded

\[
p_n(t) = \gamma \beta^{n+1} \int_0^t \sum_{k=0}^{\infty} \frac{2(n+k+1)(\alpha u)}{\beta^{k+1} \alpha u} e^{-(\gamma + \lambda + \mu)u} \, du + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} p_k e^{-(\gamma + \lambda + \mu)u} \left[ \frac{I_{m+n+k+1}(\alpha u)}{\beta^{m-n+k+1}} - \frac{I_{m+n+k+2}(\alpha u)}{\beta^{m-n+k+2}} \right] \]

\[
+ \sum_{k=0}^{\infty} p_k \beta^{-k} I_{n-k}(\alpha t) e^{-(\gamma + \lambda + \mu)t}, \quad n \geq 0,
\]

where \( I_n(\cdot) \) is the modified Bessel function of the first kind of order \( n \), \( \alpha = 2(\lambda \mu)^{1/2} \) and \( \beta = (\lambda / \mu)^{1/2} \).

In what follows, we obtain the above result in a comparatively simple manner by applying a renewal argument, used in two previous papers (see Kyriakidis (1994), (2001)) for the determination of the stationary probabilities of the simple immigration–birth–death process and the determination of the transient probabilities of the simple immigration–catastrophe process respectively.

Let \( p_{kn}(t) \), \( t \geq 0 \), be the probability that \( N(t) = n \) given that \( N(0) = k \). Clearly,

\[
p_n(t) = \sum_{k=0}^{\infty} p_k p_{kn}(t).
\]

Assume that the catastrophes are introduced at the rate \( \gamma \) even if the process is in state 0. This assumption does not change the behaviour of the process and implies that the catastrophes occur as a Poisson process with rate \( \gamma \). Let \( U_t \), \( t \geq 0 \), be the backward recurrence time, i.e. the length of time measured backwards from time \( t \) to the last catastrophe at or before \( t \). It is well known (see e.g. Cox (1962, p. 31)) that \( U_t \) is exponentially distributed with parameter \( \gamma \), censored at time \( t \), i.e. the distribution of \( U_t \) has a continuous part with density \( \gamma e^{-\gamma u} \) in the interval \((0, t)\) and a probability atom of size \( e^{-\gamma t} \) at \( t \). Therefore, conditioning on \( U_t \) we obtain

\[
p_{kn}(t) = \tilde{p}_{kn}(t)e^{\gamma t} + \int_0^t \tilde{p}_{0n}(u)e^{\gamma u} \, du,
\]

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Table 1: Transition rates in the time interval \((t, t + \delta t)\).

<table>
<thead>
<tr>
<th>Transition</th>
<th>Rate</th>
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<tbody>
<tr>
<td>(n \rightarrow n + 1)</td>
<td>(\lambda) ((n \geq 0))</td>
</tr>
<tr>
<td>(n \rightarrow n - 1)</td>
<td>(\mu) ((n \geq 1))</td>
</tr>
<tr>
<td>(n \rightarrow 0)</td>
<td>(\gamma) ((n \geq 1))</td>
</tr>
</tbody>
</table>

where \(\tilde{p}_{kn}(t) = P\{X(t) = n | X(0) = k\}\) for the simple immigration–emigration process \(\{X(t); \ t \geq 0\}\). It is known (see e.g. Cox and Smith (1961, p. 64)) that

\[
\tilde{p}_{kn}(t) = e^{-(\lambda + \mu) t} \left[ \beta^{n-k} I_{n-k}(\alpha t) + \beta^{n-k-1} I_{n+k+1}(\alpha t) + (1 - \beta^2) \beta^{2n} \sum_{j=n+k+2}^{\infty} \beta^{-j} I_j(\alpha t) \right]
\]

(4)

From (2), (3), (4) we have that

\[
p_n(t) = \int_0^t \left[ \beta^n I_n(\alpha u) + \beta^{n-1} I_{n+1}(\alpha u) + (1 - \beta^2) \beta^{2n} \sum_{j=n+2}^{\infty} \beta^{-j} I_j(\alpha u) \right] \gamma e^{-(\lambda + \mu + \gamma) u} du + \sum_{k=0}^{\infty} p_k \left[ \beta^{n-k} I_{n-k}(\alpha t) + \beta^{n-k-1} I_{n+k+1}(\alpha t) + (1 - \beta^2) \beta^{2n} \sum_{j=n+k+2}^{\infty} \beta^{-j} I_j(\alpha t) \right] e^{-(\lambda + \mu + \gamma) u}.
\]

(5)

Noting that (see e.g. Abramowitz and Stegun (1970))

\[
\frac{2(n + k + 1) I_{n+k+1}(\alpha u)}{\alpha u} = I_{n+k}(\alpha u) - I_{n+k+2}(\alpha u),
\]

it can be readily checked that the right-hand side of (1) is equal to the right-hand side of (5).

References


Yours sincerely,

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