FIBONACCI NUMBERS AND THE GOLDEN MEAN IN NATURE

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Abstract

This article introduces the Fibonacci numbers through Leonardo da Pisa's original 'breeding of rabbits' problem of 1202. Some mathematical properties of the numbers are outlined briefly. Examples of their manifestations in physics, biology and botany are presented, in particular the reconstruction of the sunflower head.

MATHEMATICAL BIOLOGY; NUCLEAR STABILITY

1. Introduction.

The numerical sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, ... first became known from the 'breeding of rabbits' problem appearing in Liber Abaci (1202) of Leonardo da Pisa (1175–1250). The Liber Abaci remained obscure until the nineteenth century when a French number theorist, Edouard Lucas (1842–1891), rediscovered it. Lucas was fascinated by the 'breeding of rabbits' problem (discussed below), and named the sequence the Fibonacci numbers; since Leonardo's father was called Bonaccio, his son would be Figlio Bonaccio, or Fibonacci for short, a nickname coined by Lucas. The properties of these numbers are so intriguing that, to Lucas, they appeared almost divine.

From the middle of the twentieth century, interest in the Fibonacci concept acquired increasing momentum, thanks to two Californian mathematicians, Brother Alfred Brousseau and the late Verner E. Hoggatt. In 1962, the Fibonacci Association was founded and, in 1963, it started publishing the Fibonacci Quarterly which celebrated its 25th anniversary in 1987.

The Fibonacci numbers form an intriguing sequence which turns up in the most unexpected areas: physics, botany, architecture, even in celebrated paintings. Fibonacci himself did not make a thorough study of the sequence; it was studied in depth by Edouard Lucas.

The celebrated 'rabbit problem' posed by Leonardo in Liber Abaci may be stated as follows. A newly born male–female pair of rabbits is kept in an enclosure to breed. This pair becomes reproductive at the age of two months, giving birth to another male–female pair; thereafter it gives birth to a male–female pair every month. Assuming that none of these pairs dies, how many pairs will

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there be at the end of one year? In Figure 1, the thick arrows indicate the
parent pair and the dotted arrows the offspring.

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4 \\
\downarrow \\
5 \\
\end{array}
\]

\[= 1 \]
\[= 2 \]
\[= 5 \]
\[= 5 \]
\[= 8 \]

Figure 1. Chart illustrating the breeding of rabbits.

We notice that the pair born at the end of the second month does not yield
progeny until the end of the fourth month. This process gives us a simple rule
for obtaining the figures in the last column. Each number is obtained by adding
the two preceding numbers in the sequence. Thus, \(1 + 2 = 3\); \(2 + 3 = 5\);
\(3 + 5 = 8\). Continuing this process, the number of rabbits at the end of twelve
months is 233, the answer to the problem. Mathematicians represent an arbi-
trary Fibonacci number by the symbol \(F_n\), where \(n\) is the position of the number
in the sequence. Fibonacci numbers satisfy the recurrence relation:

\[F_n + F_{n+1} = F_{n+2} \quad (n \geq 0).\]

2. Mathematical properties

For easy reference, the first sixteen Fibonacci numbers are given below.

\[
\begin{array}{cccccccccccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & 233 & 377 & 610 \\
\end{array}
\]

We now outline briefly some of their properties, often omitting proofs.

2.1. The sum of the squares of any two consecutive Fibonacci numbers is
also a Fibonacci number. Mathematically, \(F_n^2 + F_{n+1}^2 = F_{2n+1}\). Examples of this
relationship are:
\n
\[
\begin{align*}
1^2 + 1^2 &= 2; & 1^2 + 2^2 &= 5; & 2^2 + 3^2 &= 13; & 3^2 + 5^2 &= 34. \\
\end{align*}
\]

2.2. The product of two alternate Fibonacci numbers differs by 1 from the
square of the intermediate number, i.e. \(F_{n+1}^2 = F_n F_{n+2} \pm 1\). The following
examples illustrate this:

\[
\begin{align*}
1 \times 2 &= 1^2 + 1; & 1 \times 3 &= 2^2 - 1; & 2 \times 5 &= 3^2 + 1; \\
3 \times 8 &= 5^2 - 1; & 5 \times 13 &= 8^2 + 1; & 8 \times 21 &= 13^2 - 1. \\
\end{align*}
\]
2.3. For any four consecutive Fibonacci numbers $A$, $B$, $C$ and $D$ the following formula holds:

$$C^2 - B^2 = A \times D.$$ 

If $A = 2$, then $25 - 9 = 2 \times 8 = 16$.

2.4. With the exception of 3, every Fibonacci number that is prime has a prime subscript. For example, 233 is prime and its subscript, 13, is also prime. However, the converse is not always true. That is, a prime subscript such as 3 does not necessarily mean that the Fibonacci number is prime. The first counter-example is $F_{19} = 4181$. The subscript is prime but $4181 = 37 \times 113$.

2.5. With the exception of 0 and 1, the only square Fibonacci number is $F_{12} = 144$, which, surprisingly, is the square of its subscript. It has been proved that there is no other square Fibonacci number above 144.

2.6. Consider any two pairs of consecutive Fibonacci numbers, say 5, 8 and 13, 21. The product of the smaller numbers added to the product of the greater numbers yields a Fibonacci number. Thus $5 \times 13 + 8 \times 21 = 65 + 168 = 233$. Stated more generally, $F_n F_m + F_{n+1} F_{m+1}$ is a Fibonacci number. It can be proved quite easily using induction that $F_n F_m + F_{n+1} F_{m+1} = F_{n+m+1}$. (Property 2.1 is seen to be a special case of this formula.)

2.7. It will be seen that if the subscript of one Fibonacci number is divisible by the subscript of another, then the first Fibonacci number itself is divisible by the second. Thus $F_{15}$ (610) is divisible by $F_5$ (5) and $F_3$ (2); $F_{14}$ (377) is divisible by $F_7$ (13), and so on. Stated mathematically, if $m$ is a multiple of $n$, then $F_m$ is a multiple of $F_n$. This too can be proved by induction.

2.8. The sum of any ten consecutive Fibonacci numbers is 11 times the seventh number in the sequence, e.g.

$$5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 + 233 + 377 = 979 = 89 \times 11.$$ 

This property is true of any additive sequence regardless of the initial two numbers; we can see that the sum of the numbers $a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b$ and $21a+34b$ is $55a+88b$, which is 11 times $5a+8b$, the seventh number in the series.

2.9. Consider the ratio between two consecutive Fibonacci numbers. As their magnitudes become greater, the ratio tends to a limit, 1.618... To the first three decimal places, this ratio ($F_{n+1}/F_n$) is already 1.618 when $n = 10$.

This limiting ratio can be obtained algebraically by the following procedure. We can readily see, for example, that

$$13 = 8 + 5,$$

$$\frac{13}{8} = 1 + \frac{5}{8} = 1 + \frac{1}{\frac{8}{5}} = 1 + \frac{1}{1 + \frac{3}{5}} = 1 + \frac{1}{1 + \frac{1}{\frac{5}{3}}}.$$ 

Continuing this process we obtain the continued-fraction representation
\[
\frac{13}{8} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + 1}}}}.
\]

As \( n \) becomes infinitely large, the ratio \( \frac{F_{n+1}}{F_n} \) becomes an infinite continued fraction

\[
\frac{F_{n+1}}{F_n} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}.
\]

Let this fraction be \( x \). Clearly, from the structure of the continued fraction, we have

\[
x = 1 + \frac{1}{x}.
\]

This yields the quadratic equation

\[
x^2 - x - 1 = 0,
\]

whose roots are

\[
\phi_1 = \frac{1}{2}(1 + \sqrt{5}) = 1.618\ldots \quad \text{and} \quad \phi_2 = \frac{1}{2}(1 - \sqrt{5}) = -0.618\ldots.
\]

Thus

\[
\phi_1 + \phi_2 = 1, \quad \phi_1 \phi_2 = -1.
\]

The ratio \( \phi_1 \) (often written as \( \phi \)) is known as the golden ratio or the divine proportion.

Since \( x^{-1} = x - 1 \), the reciprocal of \( x \) is obtained by subtracting 1 from \( x \). Thus

\[
\frac{1}{1.618\ldots} = 1.618\ldots - 1 = 0.618\ldots.
\]

Indeed, the golden ratio is the limiting ratio of any additive sequence satisfying

\[
U_n + U_{n+1} = U_{n+2},
\]

regardless of the first two numbers. For example, the sequence 1, 3, 4, 7, 11, 18, 29, \ldots, called the Lucas sequence, yields the same limiting ratio.

2.10. The golden ratio gives us a formula for obtaining the \( n \)th Fibonacci number by direct calculation:

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].
\]
Proof. From our earlier equation in 2.9,
\[ \phi^2 = \phi + 1 \]
and, by induction, we can show that
\[ \phi^n = \phi F_n + F_{n-1}. \]
Thus
\[ \phi_1 F_n + F_{n-1} = \phi_1^n, \quad \phi_2 F_n + F_{n-1} = \phi_2^n, \]
from which
\[ F_n = \frac{\phi_1^n - \phi_2^n}{\phi_1 - \phi_2}. \]
Thus
\[ F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]. \]

2.11. The initial digits of Fibonacci numbers have a specific probability distribution. The probability that the first digit of a random Fibonacci number is \( n \) is given by
\[ \log_{10} \left( 1 + \frac{1}{n} \right). \]

To illustrate this law, consider the first hundred Fibonacci numbers (Table 1).

<table>
<thead>
<tr>
<th>( n )</th>
<th>Frequency of first digit ( n )</th>
<th>Observed proportion</th>
<th>( \log_{10} \left( 1 + \frac{1}{n} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>0.30</td>
<td>0.3010</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>0.18</td>
<td>0.1761</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>0.13</td>
<td>0.1249</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>0.09</td>
<td>0.0969</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>0.08</td>
<td>0.0792</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0.06</td>
<td>0.0669</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>0.05</td>
<td>0.0580</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>0.07</td>
<td>0.0512</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>0.04</td>
<td>0.0458</td>
</tr>
</tbody>
</table>

The discrepancy between the figures in the third and fourth columns would be smaller if a larger number of terms were considered. This again, is more generally true and is known as the ‘first significant digit phenomenon’. See, for instance, Pinkham (1961).

2.12. The Bessel function \( J_n(x) \) appears in many physics and engineering problems. It is defined as
\[ J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r! \Gamma(n+r+1)}. \]
Bessel functions are related to the Fibonacci numbers through formulae such as:

\[ \sum_{n=0}^{\infty} J_{n+1}(x) = e^{\frac{1}{2}x}, \quad \sum_{n=0}^{\infty} F_n J_n(x) = 0. \]

Over two thousand mathematicians around the world continue to work on the mathematical properties of the elementary-looking Fibonacci numbers, and their discoveries are published in the *Fibonacci Quarterly*. Limitation of space here will not permit reporting even a few of the very important results. One of the properties of the numerical sequence is its occurrence in the most unexpected situations. It is this aspect that particularly appealed to us, and when we examined biological phenomena, we were pleasantly surprised to encounter true Fibonacci numbers. Occurrences of these numbers have been reported in botany, zoology, biology, physics, chemistry, music and architecture, among other disciplines. We quote here a few examples from physics, botany and biology, illustrating the appearance of Fibonacci numbers in real-life situations.

3. Fibonacci numbers in physics: nuclear stability

Certain nuclei are more stable than others. A nucleus is essentially a collection of protons and neutrons. The number of protons, \( Z \), is the atomic number, while the number of neutrons, \( N \), is the neutron number. Nuclei whose \( Z \) or \( N \) numbers take on certain values are exceptionally stable. These values are known as 'magic numbers'. They are 2, 8, 14, 20, 28, 50, 82 and 126. It will be seen that if these numbers are divided by 10 and expressed to the nearest whole number, the result is the Fibonacci sequence shown in Table 2.

<table>
<thead>
<tr>
<th>Magic numbers ( x )</th>
<th>( x/10 )</th>
<th>Nearest whole number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>1.4</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>2.0</td>
<td>2</td>
</tr>
<tr>
<td>28</td>
<td>2.8</td>
<td>3</td>
</tr>
<tr>
<td>50</td>
<td>5.0</td>
<td>5</td>
</tr>
<tr>
<td>82</td>
<td>8.2</td>
<td>8</td>
</tr>
<tr>
<td>126</td>
<td>12.6</td>
<td>13</td>
</tr>
</tbody>
</table>

An explanation of these magic numbers was finally given by Maria Goeppert Mayer and Hans Jensen, who postulated the shell model of the nucleus.

4. Fibonacci numbers in biology

4.1. Ridges on the horns of bighorn sheep. The impressively spiralling horns of the bighorn sheep show evidence of the presence of the Fibonacci system. Professor Richard Goss of the Biology Faculty, University of Rhode Island, discovered that the horns display sets of 13 ridges and furrows; 13 is a Fibonacci number. In Figure 2, the five-year-old horn of a bighorn sheep is shown. The very tip of the horn, which is the first-year growth, has no ridges or grooves on
its surface as the animal was immature. But the portion for every subsequent annual growth of horn shows 13 clear ridges between the alternating grooves. According to Goss (1969) the grooves represent the short period when the animal undergoes pain during its estrous period, when normal growth is reduced. As there are 13 lunar months in the year, which synchronise with the oestral cycle of the animal, 13 ridges (or grooves) become visible on each horn. The portions of horn developed between the years can be distinguished easily because of a deeper groove after every 13 ridges.

4.2. *Golden mean of the human body*. Leonardo da Vinci found that the total height of the human body and the height from the toes to the navel are roughly in the golden ratio. We have confirmed this by measuring 207 students at the Pascal Gymnasium in Munster, West Germany, where the almost perfect value of 1.618 was obtained for the average. This value held for both girls and boys of similar age. However, similar measurements of 252 young men in Calcutta, India, gave a slightly different value of 1.615. The tallest and shortest subjects in the German sample differed slightly in body proportions, but no such difference was noted among the Indians in the Calcutta sample. Figure 3 illustrates the human body with various organs indicated.

5. *Fibonacci numbers in botany*

5.1. *Foliar spirals in palms*. About 2700 species of palms constitute the family of Arecales. Most of them possess very large leaves which are produced one after another (spiral phyllotaxy) on a stout stem. Any two consecutive leaves subtend the Fibonacci angle of approximately 137.5°. (The ratio of the remaining angle of 222.5°, to complete one full revolution, to the Fibonacci angle of 137.5° is the familiar golden ratio, 1.618 ... ) Because of this arrangement, the leaves appear spirally and the phyllotaxy may be 2/3, 3/5, 5/8, 8/13, 13/21, ..., or 2/5, 3/8, 5/13, 8/21, 13/34, 21/55, and so on.
Different species of palms display different numbers of spirals, and their numbers always match a Fibonacci number. For example, in the areca palm (*Areca catechu*) or the ornamental *Ptychosperma macarthurii* palm, only a single foliar spiral is discernible, while in the Indonesian sugar palm (*Arenga pinnata*), two spirals are visible. In the palmyra palm (*Borassus flabellifer*) or the talipot palm (*Corypha elata*), three clear spirals are visible. The coconut (*Cocos nucifera*) has five leaf spirals, while the African oil palm (*Elaeis guineensis*) or the sugar date palm (*Phoenix sylvestris*) shows eight spirals. It is also possible to detect in some of these stems five spirals veering opposite to the eight spirals. In the Canary Island date palm (*Phoenix canariensis*), a maximum of thirteen spirals can be made out. In some exceptionally stout trunks of *P. canariensis*, twenty-one spirals could also be traced. These numbers (1, 2, 3, 5, 8, 13, 21) are Fibonacci numbers (Figure 4). It is interesting to realise that there is no palm that bears 4, 6, 7, 9, 10, 11, 12 or 14 spirals. From the schematic representation of a palm crown bearing 48 leaves (Figure 5) all these sets of leaf spirals can be made out.

5.2. *The ray florets on the sunflower head*. The flower head or capitulum of the sunflower has two kinds of flowers. The irregular, petal-like flowers which occupy the periphery of the disc are called the ray florets, which are either female or sterile flowers. The numerous regular flowers developing on the flat surface of the head are called the disc florets. From the arrangement of disc florets, spirals or arcs can be made out, and their numbers have relevance to the Fibonacci numbers. The number of petal-like ray florets varies per capitulum according to species and according to the diameter of the disc. For example, *Tridax procumbens* has five ray florets, *Cosmos* has eight florets, *Rudbekia* has thirteen ray florets, and so on. The small-sized capitulum of a sunflower (*Helianthus annuus*) may have thirteen ray florets, and this number will increase as the diameter of the disc increases. Two undergraduate students at the Indian Statistical Institute, Calcutta, examined about a thousand heads of a small-sized...
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Figure 4. Trunks of palms showing 1, 2, 3, 5, 8 and 13 foliar spirals.

Figure 5. Schematic representation of a palm crown bearing 48 leaves.

and profusely branching variety of sunflower, and the flower heads were graded according to the number of ray florets per head. The graph in Figure 6 represents the data in percentages. It is impressive that the mode is at 21, which is a Fibonacci number. Also there is a smaller mode at 13, which is again a Fibonacci number. In another sample of larger heads, the modes happen to be at 21, 34 and 55, all Fibonacci numbers (Majumder and Chakravarti (1976)).
5.3. **Patterns on the sunflower head.** Numerous publications on the properties of Fibonacci numbers and their application have appeared, especially during the past forty years (Alfred (1965), Bicknell and Hoggatt (1972), Huntley (1970), Vorob'ev (1961)). The present authors have also had some share in these developments, especially on the application of Fibonacci numbers in physics and biological specimens (Davis, B. S. (1972), (1976), Davis, B. S. and Hoggatt (1976), Davis, T. A. (1968), (1970), (1971), (1977), Davis, T. A. and Altevogt (1979), Davis, T. A. and Mathai (1969)).

One of the problems that has attracted the attention of mathematicians, engineers and biologists during the past hundred years is how to model the construction of the head of a sunflower, with its display of impressive arcs and spirals. But lack of basic knowledge of the growth principles of plants prevented mathematicians from completing the study of the problem (Bergamini (1965), Gardner (1969), Kilmer (1971)). Most biologists, not understanding the mathematics involved in the arrangement of the flowers, could not offer a correct explanation for the floral arrangement of the sunflower head.

Relying mainly on Fibonacci properties, we were able to reconstruct the sunflower head (Mathai and Davis (1974)). The procedure consisted of assigning the Fibonacci angle of 137.507...° between any two consecutive individual flowers (florets) and controlling the logarithmic scatter of the floral positions. The florets are formed one at a time on the highly compressed stem which flattens out into the disc. The disc widens as more and more florets are differentiated, and the older ones move away from the growing point (the central region), while the younger ones are distributed around the central point. A flower primordium is differentiated on a side of the stem apex, and the subsequent florets are generated at a fast rate with a constant time interval between any two consecutive individuals (Figure 7). As the flowers become differentiated, the tip of the meristemic axis rotates so that the older florets are seen to move away from the growing point in logarithmic spirals that approximate an Archimedean spiral. Moreover, among any two consecutive florets, the younger
Figure 7. Emergence of floral primordia of sunflower which follows an Archimedean spiral.

Figure 8. A. Reconstructed sunflower head with uniform fruitlets; B. Giant capitulum of sunflower where the entire surface is filled with uniform fruitlets.

One starts differentiating from the axis when the older one is at an angle $\phi_1$, so that $(2-\phi_1)/\phi_1$ forms the golden ratio 1.618...

This process continues until the development of the flower head is complete. With favourable environmental conditions, the individual flowers expand at a uniform rate over time along with the simultaneous expansion of the disc (Figure 8A). Using our method, we were able to maintain the size of the fruitlets uniform throughout the disc and also avoid any empty space at the centre. The reconstructed head matches the natural head shown in Figure 8B.

6. Concluding remarks

Fibonacci numbers appear in many unexpected situations. Numerous manifestations of Fibonacci sequences may be found in various natural phenomena, perhaps most strikingly among plants and their organs. By employing certain
Fibonacci properties, some major problems have been solved, such as the tenth problem of David Hilbert and the century-old problem of reconstructing the head of the sunflower.

References


