Generating a conic
River lengths
We need some new functions
A complex series of voyages of discovery

A magazine for students and teachers of mathematics in schools, colleges and universities
MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of Mathematical Spectrum is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in Mathematical Spectrum deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing. The copyright of all published material is vested in the Applied Probability Trust.

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**From the Editor**

**Why mathematics?**

That is a question many young people ask as they decide which subjects to study at school or at university, or contemplate a career. The answers given will be as varied as are people themselves: enjoyment, ‘it’s my best subject’, it’s good for a career, and so on. But is there a higher reason inherent in the subject itself? Does it make you a ‘better person’?

It can be argued that studying any subject has a similar effect, whether in the arts or sciences. Education is in some way a civilising influence. You are perhaps less likely to be attacked in the street by a university graduate than by someone who dropped out of school at 16, although the reasons for this may be found further back. And the graduate may have more sophisticated ways of doing you ill!

But is there something special about the study of mathematics? After all, it is the subject which, in essence, is based on strict logical thought. The mathematical method is to start with a set of axioms, that one hopes are consistent, and make deductions from them, the theorems, using rules of logic. Can this method be applied in other areas? We are told that employers like to take on mathematics graduates, not just for their mathematical knowledge but also because of the processes of logical thought that they have acquired and which they can then apply to the diverse problems that confront humankind. There is surely some truth in this. But how much? And is it the whole truth?

I was pulled up sharply by a book that came for review, *Mathematicians Under the Nazis* by Sanford Segal. This is not a book for the general reader; it is too encyclopaedic. It is a comprehensive study of the subject for the serious student of mathematical history. Professional university mathematicians will recognize many household names that are now part of the nomenclature of mathematics. But it is fascinating to see how mathematicians in Germany in the 1930s reacted to Nazism. Did their logical training help them to assess the situation they faced and to react to it? It would be hard to argue that from this study. Reactions were as varied as the people themselves. They were probably no worse than students of other disciplines. But were they any better? Probably not. There seems to be something deeper that is involved. I must leave you to decide what that might be. Perhaps we also need to ask another question: how would I have reacted?


---

**Problem.** How to evaluate $r^x$ $(r, x \in \mathbb{R})$ where $x$ is so large that it would cause an overflow error in a calculator.

First consider

$$r^y = 10^y .$$

Now, taking logs to base 10, we have $x \log r = y$. Therefore,

$$r^x = 10^{x \log r} .$$

Now, let $f = \text{frac}(x \log r)$ and $i = \text{int}(x \log r)$. Then,

$$r^x = 10^f \cdot 10^i .$$

For example, evaluate $z = 5.35^{200}$. Here,

$$f = \frac{\text{frac}(200 \log 5.35)}{200 \log 5.35} = 0.6707564 ,$$

$$i = \text{int}(200 \log 5.35) = 145 ,$$

$$10^f = 4.68550494 .$$

Therefore, $z \approx 4.6855 \times 10^{145}$. Alternatively,

$$5.35^{100} = 6.84507486 \times 10^{72} ,$$

$$5.35^{200} = 6.84507486^2 \times (10^{72})^2$$

$$= 46.85504984 \times 10^{144} .$$

**Seyamack Jafari**

Razi Petrochemical Complex,
Bandar, Imam, Khozestan, Iran.
Generating a Conic

GUIDO LASTERS and HUGO STAELENS

We start with a conic, say an ellipse, and a point P on the ellipse. Draw the tangent to the ellipse at P and the diameter OP, where O is the centre of the ellipse. Then every chord A0A1 parallel to the tangent at P is bisected by OP; see figure 1.

The same is true of a hyperbola and also of a parabola, where now O has moved to infinity and the diameter OP becomes the straight line through P parallel to the axis of symmetry of the parabola; see figure 2.

To prove this property for a parabola, let the equation of the parabola be \( y = ax^2 \) and let P be the point \((k, ak^2)\). Then the chord has slope 2ak and so has equation of the form

\[ y = 2akx + c. \]

This meets the parabola in points A0 and A1 whose x-coordinates \(x_1, x_2\) are solutions of the equation

\[ 2akx + c = ax^2 \]

or

\[ ax^2 - 2akx - c = 0, \]

so the sum of the roots is

\[ x_1 + x_2 = 2k. \]

The midpoint of A0A1 thus has x-coordinate

\[ \frac{x_1 + x_2}{2} = k \]

and so lies on the line \(x = k\), as claimed. We leave readers to prove similar properties for an ellipse and a hyperbola — see problem 36.5 in the previous issue.

We can use this property to generate a sequence of points on a conic. We first designate two distinct directions by means of two straight lines \(m, n\). Now designate a point O and distinct straight lines \(r\) and \(s\) through O, with \(r\) not parallel to \(m\) and \(s\) not parallel to \(n\). Start with a point A0 not on \(r\) and reflect it in \(r\) in a direction parallel to \(m\) to give the point A1. Thus, A0A1 is parallel to \(m\) and \(r\) bisects A0A1. Now reflect A1 in \(s\) in a direction parallel to \(n\) to give the point A2. Now reflect A2 in \(r\) in a direction parallel to \(m\) to give the point A3. And so on. The sequence of points A0, A1, A2, A3, ... will lie on a conic; see figure 3.

When \(r\) and \(s\) are parallel, the conic generated in this way will be a parabola. We can determine its equation relative to suitable axes as follows. Choose axes so that the y-axis is parallel to \(r\) and \(s\) and lies midway between them, so that \(r\) and \(s\) will have equations

\[ x = k \quad \text{and} \quad x = -k \quad \text{for some} \quad k \neq 0. \]

The x-axis is chosen to be any line perpendicular to these. Denote by \(m, n\) the slopes of the two designated directions for the reflections, and let A0 be the point \((x_0, y_0)\). Let the parabola generated by our construction have equation

\[ y = ax^2 + bx + c. \]

The straight line A0A1 has slope \(m\) and is parallel to the tangent to the parabola at the point with x-coordinate \(k\), so that

\[ m = 2ak + b. \]
Adding the four inequalities, we obtain that

\[ n = -2ak + b. \]

Hence,

\[ m - n = 4ak, \quad m + n = 2b, \]

so that

\[ a = \frac{m - n}{4k}, \quad b = \frac{m + n}{2}. \]

The point \( A_0 \) lies on the parabola, so that

\[ y_0 = ax_0^2 + by_0 + c, \]

and so

\[ c = y_0 - ax_0^2 - bx_0. \]

Guido Lasters (Tienen, Belgium) teaches in Leuven and Hugo Staelens (Eeklo, Belgium) is retired.

---

A Simple Proof of Euler’s Inequality in Space

ZHANG YUN

Let \( R \) be the radius of the circumscribed sphere of a tetrahedron and let \( r \) be the radius of the inscribed sphere of the tetrahedron. Then Euler’s famous inequality in space states that

\[ R \geq 3r. \]

We give here a simple proof of this inequality.

Let \( O \) be the circumcentre of the tetrahedron \( A_1A_2A_3A_4 \). Let \( s_k \) (\( k = 1, 2, 3, 4 \)) denote the area of the face opposite the vertex \( A_k \), let \( h_k \) denote the distance from \( A_k \) to its opposite face, and let \( d_k \) denote the distance from the point \( O \) to the face opposite \( A_k \). Then

\[ OA_k + d_k \geq h_k, \]

and so

\[ R + d_k \geq h_k. \]

Thus,

\[ s_k R + s_k d_k \geq s_k h_k. \]

Adding the four inequalities, we obtain that

\[ R(s_1 + s_2 + s_3 + s_4) + s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4 \]

\[ \geq s_1h_1 + s_2h_2 + s_3h_3 + s_4h_4. \]

Let \( V \) denote the volume of the tetrahedron \( A_1A_2A_3A_4 \). Then

\[ V = \frac{1}{6}s_k h_k = \frac{1}{6}(s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4), \]

so

\[ R(s_1 + s_2 + s_3 + s_4) + 3V \geq 4 \times 3V, \]

from which it follows that

\[ R(s_1 + s_2 + s_3 + s_4) \geq 9V. \]

Since

\[ V = \frac{R}{3}(s_1 + s_2 + s_3 + s_4), \]

this gives

\[ R(s_1 + s_2 + s_3 + s_4) \geq 9 \times \frac{R}{3}(s_1 + s_2 + s_3 + s_4). \]

Thus,

\[ R \geq 3r, \]

so the inequality (1) is proved.

In two dimensions rather than three, if \( R \) is now the radius of the circumscribed circle of a triangle and \( r \) the radius of the inscribed circle, then, by a similar argument, \( R \geq 2r \).

Zhang Yun is a senior teacher (associate professor) of mathematics at the first middle school of Jinchang City, Gan Su Province in China. He has published over 100 papers. His research interests include elementary number theory, algebraic inequalities and geometric inequalities.
Distribution of River Lengths and the Total Length of Rivers

A. TAN and W. SHENG

1. Introduction

Rivers are among mankind’s most vital assets. The earliest civilizations were located near rivers. Today, the greatest cities, as well as the densest population centres, are situated in the major river basins. In high-school geography classes, students learn about the longest rivers in the world. In this article we examine the relationship between the ranks and the lengths of rivers.

Lists of the longest rivers are found in atlases and handbooks. They differ slightly from one another and are invariably incomplete. One of the longest lists (containing 75 longest rivers) is found in reference 1 and is reproduced here in table 1. The ranks and lengths of the rivers (in kilometres) and the continents in which they run are also given.

<table>
<thead>
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<th>Rank</th>
<th>River</th>
<th>Continent</th>
<th>Length (km)</th>
</tr>
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<td>6671</td>
</tr>
<tr>
<td>2</td>
<td>Amazon</td>
<td>South America</td>
<td>6300</td>
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<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>Mississipi–Missouri</td>
<td>North America</td>
<td>6019</td>
</tr>
<tr>
<td>5</td>
<td>Ob–Irtysh</td>
<td>Asia</td>
<td>5411</td>
</tr>
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<td>75</td>
<td>Northern Dvina</td>
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<td>1302</td>
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2. Length as a power function of rank

A visual examination of the data indicates that the latter are likely to follow a power function law, where the length of a river \( L \) as a function of its rank \( N \) is given by

\[
L = CN^{\gamma - 1} \tag{1}
\]

(see reference 2, pp. 161–164). Here \( \gamma \) is the shape parameter \( (1 > \gamma > 0) \) and \( N \in \{1, 2, \ldots, n\} \), the number of data points (\( n = 75 \) in our case). It is convenient to rewrite (1) as

\[
L = C N^{\beta}, \tag{2}
\]

with \( \beta = 1 - \gamma < 1 \). Taking the logarithm of both sides of (2), we get

\[
\log C - \beta \log N = \log L. \tag{3}
\]

Multiplying both sides of (3) by \( N \) then gives

\[
N \log C - \beta N \log N = N \log L. \tag{4}
\]

The normal equations are obtained by summing (3) and (4) over the \( n \) data points \( N \):

\[
n \log C - \beta \sum_{N=1}^{n} \log N = \sum_{N=1}^{n} \log L \tag{5}
\]

and

\[
\log C \sum_{N=1}^{n} N - \beta \sum_{N=1}^{n} N \log N = \sum_{N=1}^{n} N \log L. \tag{6}
\]

For a least-squares fit of the data points, the constants \( C \) and \( \beta \) are obtained by elimination from (5) and (6):

\[
\beta = \frac{\sum_{N=1}^{n} N \sum_{N=1}^{n} \log L - \frac{n}{N} \sum_{N=1}^{n} N \log N \sum_{N=1}^{n} \log L}{n \sum_{N=1}^{n} N \log N - \sum_{N=1}^{n} N \sum_{N=1}^{n} \log N} \tag{7}
\]

and

\[
C = \exp \left[ \frac{\beta \sum_{N=1}^{n} \log N + \sum_{N=1}^{n} \log L}{n} \right]. \tag{8}
\]

Applying (7) and (8) to the data from table 1 gives the estimates \( \hat{C} = 13,680.68 \) and \( \hat{\beta} = 0.4993653 \). Figure 1 shows the power function fitted to river lengths thus obtained. It can be observed that the function overestimates the river lengths for \( N = 1, \ldots, 9 \), underestimates them for \( N = 11, \ldots, 50 \) and again overestimates them for \( N > 60 \) near the tail end of the curve. A plausible reason for the overestimation of the first nine lengths is that the lengths of the longest rivers are limited by the sizes of the continents in which they run. The areas of the continents \( (S) \) are taken from reference 3 and entered in table 2. Also shown in the table are \( \sqrt{S} \) (which is a measure of the linear dimensions of the continents) and the lengths of the longest river on each continent. The fact that the two are comparable indicates that the lengths of the longest rivers are indeed restricted by the sizes of the continents.
Table 2. The areas of the continents compared with the lengths of their longest rivers and how many of the longest rivers are in each continent.

<table>
<thead>
<tr>
<th>Continent</th>
<th>Area, S (km²)</th>
<th>(\sqrt{S}) (km)</th>
<th>Length of longest river (km)</th>
<th>Number of the longest 10 rivers in continent</th>
<th>Number of the longest 75 rivers in continent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asia</td>
<td>43 608 000</td>
<td>6604</td>
<td>6276</td>
<td>6</td>
<td>21</td>
</tr>
<tr>
<td>Africa</td>
<td>30 335 000</td>
<td>5508</td>
<td>6671</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>North America</td>
<td>25 349 000</td>
<td>5035</td>
<td>6019</td>
<td>1</td>
<td>24</td>
</tr>
<tr>
<td>South America</td>
<td>17 611 000</td>
<td>4197</td>
<td>6300</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>Europe</td>
<td>10 498 000</td>
<td>3240</td>
<td>3531</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Australasia</td>
<td>8 923 000</td>
<td>2987</td>
<td>3718</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2 shows that of the 10 longest rivers in the world, 6 (or 60%) are in Asia (the largest continent), 2 are in Africa (the second largest continent) with one each in North and South America (the third and fourth largest continents). The longest rivers of the smallest continents, Australasia and Europe, rank 17th and 18th respectively. Interesting too is the fact that, of the 75 longest rivers, 5 are in Africa and only 2 are in Australasia. Australia is by far the driest continent, and northern Africa has the largest desert in the world.

3. Length as an exponential function of rank

The river length \(L\) may also be expressed as an exponential function of the rank \(N\):

\[
L = Ae^{-\alpha N},
\]

where \(N \in \{1, 2, \ldots\}\).

The two constants \(A\) and \(\alpha\) can be estimated as follows: take logarithms of both sides of (10) and sum over the \(n\) (= 75) data points; then obtain \(A\) and \(\alpha\) from the normal equations by elimination:

\[
\log A = \frac{\sum_{N=1}^{n} \log L}{n} \left( \frac{1}{N} \log N \right) - \frac{\sum_{N=1}^{n} N \log L}{n} \frac{N^2}{\left( \sum_{N=1}^{n} N \right)^2 - n \sum_{N=1}^{n} N^2} \sum_{N=1}^{n} \log L \sum_{N=1}^{n} \frac{N^2}{N^2}
\]

and

\[
\alpha = \frac{n \sum_{N=1}^{n} \log L - \sum_{N=1}^{n} \log L \sum_{N=1}^{n} N}{\left( \sum_{N=1}^{n} N \right)^2 - n \sum_{N=1}^{n} N^2}.
\]

Upon substituting our data, we obtain \(L_{75} \approx 209,979\) km. This compares very favourably with the actual value of 209,957 km obtained by summing the lengths in table 1.

Figure 2. The exponential function fitted to ranked river lengths. The length \(L = Ae^{-\alpha N}\) of the river is plotted against its rank \(N\) using the estimates \(\hat{A} = 5235.338\) and \(\hat{\alpha} = 0.0188547\).
Substituting the data from table 1 gives the estimates \( \hat{A} = 5235.338 \) and \( \hat{\alpha} = 0.0188547 \). Figure 2 shows the exponential function fitted to river lengths. This underestimates the river lengths for \( N = 1, \ldots, 6 \) and overestimates them for \( N = 18, \ldots, 36 \).

For the exponential function, the total length of the 75 longest rivers can be approximated by the integral

\[
L_n \approx \int_1^n A e^{-\alpha N} dN = \frac{A}{\alpha} \left[ e^{-\alpha} - e^{-n\alpha} \right].
\]

Upon substitution, we get

\[
L_{75} \approx 204951 \text{ km},
\]

which is not too distant from the actual value of 209957 km. The exponential function furnishes a rough finite value for the total length of all rivers, while the power function gives an infinite total length.

### References


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We Need Some New Functions

MARGEN ÇUKO and PAUL BELCHER

We know that

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x,
\]

but what is

\[
\int \frac{1}{\sqrt{1-x^4}} \, dx
\]

equal to? We cannot solve this integral with elementary functions, so let us try and generalize circular functions.

![Figure 1](image).

1. **The circular function**

Consider the circle given in polar co-ordinates by the equation \( r = \sin \theta \) (where \( r \) is allowed to be negative). This is the circle with centre \((0, \frac{1}{2})\) and radius \(\frac{1}{2}\). The length \(t\) of the arc OA of the circle in figure 1 given by the arc-length formula

\[
\int \sqrt{1+r^2(d\theta/dr)^2} \, dr
\]

is

\[
t = \int_0^R \sqrt{1 + \frac{r^2}{\cos^2 \theta}} \, dr = \int_0^R \frac{1}{\sqrt{1-r^2}} \, dr = [\arcsin r]_0^R = \arcsin R,
\]

where \(R\) is the length of the chord OA. Since \(t = \arcsin R\), \(R = \sin t\). The formula \(t = \arcsin R\) is originally defined for \(R \in [0, 1]\) and \(t \in [0, \pi/2]\) and then extended to \(R \in [-1, 1]\) and \(t \in [-\pi/2, \pi/2]\) by taking \(\arcsin -R\) as an odd function and using \(\arcsin(-R) = -\arcsin R\). We consider the arc length to be positive when proceeding anticlockwise from O and negative when proceeding clockwise from O.

The domain of the function \(\sin\) can then be extended to \([-\pi, \pi]\) using the symmetry rules

\[
\sin \left( \frac{\pi}{2} + t \right) = \sin \left( \frac{\pi}{2} - t \right)
\]
and 
\[ \sin \left( -\frac{\pi}{2} - t \right) = \sin \left( -\frac{\pi}{2} + t \right) \quad \text{for} \quad t \in \left[ 0, \frac{\pi}{2} \right] \]

and then further extended to \( \mathbb{R} \) using a periodicity of \( 2\pi \).
Basic graphs of \( \sin t \) and \( \arcsin R \) are given in figure 2.

2. The lemniscate function

Now consider in an analogous way the lemniscate curve \( r^2 = \sin 2\theta \). This curve is undefined for \( \theta \in (\pi/2, \pi) \) and \( \theta \in (3\pi/2, 2\pi) \), whereas for \( \theta \in [0, \pi/2] \) and \( \theta \in [\pi, 3\pi/2] \) there are two values for \( r \), a positive and a negative one. The length \( t \) of the arc OA of the lemniscate curve in figure 3 is given by

\[
t = \int_{0}^{R} \sqrt{1 + r^2 \frac{r^2}{\cos^2 2\theta}} \, dr = \int_{0}^{R} \frac{1}{\sqrt{1 - r^4}} \, dr.
\]

If we define a new function \( \text{arcsinl} \) by

\[
\text{arcsinl} R = \int_{0}^{R} \frac{1}{\sqrt{1 - r^4}} \, dr,
\]

then \( t = \text{arcsinl} R \) for \( R \in [-1, 1] \) and \( t \in [-w/2, w/2] \), where \( w \) is the length of the lemniscate leaf in the first quadrant. Again, initially this definition holds for \( R \in [0, 1] \) and \( t \in [0, w/2] \) and can then be extended to the larger domain and range by making it an odd function. Denoting the inverse function of \( \text{arcsinl} \) by \( \text{sinl} \), for the sine lemniscate we have \( R = \sin t \). Now we consider the period of \( \text{sinl} \).
For the usual, circular, sine the period is \( 2\pi \) where \( \pi \) is the circumference of the circle in figure 1 that is traced out twice as \( \theta \) goes from 0 to \( 2\pi \). As \( w \) is the arc length of one of the lemniscate leaves, we have

\[
\frac{1}{2} w = \int_{0}^{1} \frac{1}{\sqrt{1 - r^4}} \, dr
\]

(which is approximately 1.311) and the length of the whole lemniscate will be \( 4w \) as \( \theta \) goes from 0 to \( 2\pi \) (since the shape in figure 3 is traced out twice). The function \( \text{sinl} \) can then have its domain extended by the symmetry rules

\[
\sinl \left( w - t \right) = \sinl \left( -\frac{w}{2} + t \right)
\]
and

\[
\sinl \left( -\frac{w}{2} - t \right) = \sinl \left( -\frac{w}{2} + t \right) \quad \text{for} \quad t \in \left[ 0, \frac{w}{2} \right]
\]

and then extended again to make it periodic with period \( 2w \).

In the circle, \( R \) is negative when \(-\pi/2 < t < 0\); in the lemniscate, \( R \) is negative when \(-w/2 < t < 0\). Using this symmetry we have the following equalities:

\[
\sinl (w - t) = \sinl (t),
\]
\[
\sinl (w + t) = -\sinl (t),
\]
\[
\sinl (2w - t) = \sinl (-t) = -\sinl (t).
\]

Since \( \cos t = \sin(\pi/2 - t) \), by analogy we will define the cosine lemniscate function \( \text{cosl} \) by

\[
\cosl (t) = \sinl \left( \frac{w}{2} - t \right).
\]
This gives
\[ \cos(w + t) = - \cos(t), \]
\[ \cos(w - t) = - \cos(t), \]
\[ \cos(2w - t) = \cos(-t) = \cos(t). \]

If we set \( u = \cos(s) \) for some \( s \), then \( w = \sin(w/2 - s) \).

Using the notation \( s = \arccos(u) \), as \( w/2 - s = \arcsin u \), we then have \( \arccos(u) = w/2 - \arcsin(u) \). So
\[
\arccos(u) = \int_0^1 \frac{1}{\sqrt{1 - r^2}} \, dr - \int_0^u \frac{1}{\sqrt{1 - r^2}} \, dr
\]
\[
= \int_u^1 \frac{1}{\sqrt{1 - r^2}} \, dr. \tag{1}
\]

Now we attempt to find an analogue of the equality \( \sin^2 t + \cos^2 t = 1 \). Let us apply the substitution given by \( r^2 + y^2 + r^2 y^2 = 1 \) to the integral \( \int_0^R (1/\sqrt{1 - r^2}) \, dr \) (which, by definition, is \( \arcsin R = t \)), to give
\[
\int_1^{(1-R^2)/(1+R^2)} \frac{1}{\sqrt{1 - (1 - y^2)/(1 + y^2)^2}} \times \frac{-(y + y((1 - y^2)/(1 + y^2))))}{(1 - y^2)/(1 + y^2)(1 + y^2)} \, dy.
\]
This simplifies to
\[
\int_1^{(1-R^2)/(1+R^2)} \frac{1}{\sqrt{1 - y^4}} \, dy,
\]
which, from (1), is equal to
\[
\arccos \left( \frac{1 - R^2}{\sqrt{1 + R^2}} \right).
\]
So
\[
t = \arcsin(R) = \arccos \sqrt{\frac{1 - R^2}{1 + R^2}}
\]
and, hence,
\[
R = \sin t \quad \text{and} \quad \sqrt{\frac{1 - R^2}{1 + R^2}} = \cos t,
\]
giving the identity
\[
\cos^2(t) + \sin^2(t) + \cos^2(t) \sin^2(t) = 1. \tag{2}
\]

Now let \( y = \sin x \). Then \( x = \arcsin y \) and differentiating with respect to \( y \) gives
\[
1 = \frac{1}{\sqrt{1 - y^4}} \frac{dy}{dx}.
\]
Hence \( \frac{dy}{dx} = \sqrt{1 - y^4} \), so
\[
\frac{d}{dx} \sin x = \sqrt{1 - \sin^4 x} = \frac{2 \cos x}{1 + \cos^2 x} = \cos x \left(1 + \sin^2 x \right).
\]

Similarly, giving the correct sign,
\[
\frac{d}{dx} \cos x = - \sqrt{1 - \cos^4 x} = \frac{-2 \sin x}{1 + \sin^2 x} = - \sin x \left(1 + \cos^2 x \right).
\]

Now \( \sin x \) and \( \cos x \) are periodic functions, with period \( 2\pi \), from \( \mathbb{R} \) to \([-1, 1]\). So, for \( \arcsin t \) and \( \arccos t \) to be functions, we need to choose principal values for them. For \( \arcsin \) these are between \(-w/2\) and \( w/2 \) and for \( \arccos \) these are between 0 and \( w \). So
\[
\arcsin: [-1, 1] \to \left[-\frac{w}{2}, \frac{w}{2}\right],
\]
\[
\arccos: [-1, 1] \to [0, w]\text{.}
\]

The graphs of \( R = \sin t \) for \( t \in [0, w/2] \) and \( t = \arcsin R \) for \( R \in [0, 1] \) are given in figure 4.

3. The trefoil function

It now seems natural to look at the ‘rose with three petals’ given by \( r^3 = \sin 3\theta \) (where again we will allow \( r \) to be negative).

The length \( t \) of the arc OA in figure 5 is given by
\[
\int_0^R \sqrt{1 + r^2 \cos^2 \theta} \, dr = \int_0^R \frac{1}{\sqrt{1 - t^6}} \, dr.
\]

We will call this expression \( \arcsin_3 R \). Denoting the inverse of this function by \( \sin_3 \), we have \( R = \sin_3 t \). Again we have \( R \in [-1, 1] \) after we have extended from \( R \in [0, 1] \).
for an odd function and negative and positive arc lengths for $t$. Let $w_3$ be the length of the leaf of the trefoil in the first quadrant. Then

$$w_3 = \int_0^1 \frac{1}{\sqrt{1 - r^6}} \, dr$$

(which is approximately 1.214) and the length of the whole curve will be $6w_3$ as $\theta$ goes from 0 to $2\pi$. If we set $u = \cos_3(s)$ for some $s$, then $u = \sin_3(w_3/2 - s)$. Hence $w_3/2 - s = \arcsin_3 u$ and $\arccos_3(u) = w_3/2 - \arcsin_3(u)$. So

$$\text{arccos}_3 u = \int_0^1 \frac{1}{\sqrt{1 - r^6}} \, dr - \int_0^w \frac{1}{\sqrt{1 - r^6}} \, dr$$

$$= \int_0^1 \frac{1}{\sqrt{1 - r^6}} \, dr.$$  \hfill (3)

Then $\text{arccos}_3 u = -\int_0^w (1/\sqrt{1 - r^6}) \, dr$ gives

$$\frac{d}{dx} \arccos_3 x = \frac{-1}{\sqrt{1 - x^6}}.$$  

Now $\sin_3 x$ and $\cos_3 x$ are periodic functions, with period $2w_3$, from $\mathbb{R}$ to $[-1, 1]$. So, for $\arcsin_3 t$ and $\arccos_3 t$ to be functions, we need to choose principal values for them. For $\arcsin_3$ these are between $-w_3/2$ and $w_3/2$; for $\arccos_3$ they are between 0 and $w_3$. So

$$\arcsin_3 : [-1, 1] \to \left[ -\frac{w_3}{2}, \frac{w_3}{2} \right],$$

$$\arccos_3 : [-1, 1] \to [0, w_3].$$

To find the connection between $\sin_3 t$ and $\cos_3 t$, we apply the substitution $r^2 + y^2 + 2r^2y^2 = 1$ to the integral

$$\int_0^R (1/\sqrt{1 - r^6}) \, dr \quad \text{(which is $\arcsin_3 R = t$), to give}$$

$$\int_1^{\sqrt{(1-R^2)/(1+2R^2)}} \frac{1}{\sqrt{1 - ((1-y^2)/(1+2y^2))^3}}

\times \frac{-(y(1+2y^2) + 2y(1-y^2))}{\sqrt{(1-y^2)/(1+2y^2)(1+2y^2)}} \, dy.$$  

This simplifies to

$$\int_1^{\sqrt{(1-R^2)/(1+2R^2)}} \frac{1}{\sqrt{1 - y^6}} \, dy,$$

which, using (3), is equal to $\arccos_3 \sqrt{(1-R^2)/(1+2R^2)}$. So

$$t = \arcsin_3 R = \arccos_3 \sqrt{\frac{1-R^2}{1+2R^2}}.$$  

Now, $R = \sin_3 t$ and $\sqrt{(1-R^2)/(1+2R^2)} = \cos_3 t$, giving

$$\cos_3^2 t + \sin_3^2 t + 2 \sin_3^2 t \cos_3^2 t = 1. \quad (4)$$

Let $y = \sin_3 x$. Then $x = \arcsin_3 y$ and differentiating with respect to $x$ gives

$$1 = \frac{1}{\sqrt{1 - y^6}} \frac{dy}{dx}.$$
Hence, \( \frac{dy}{dx} = \sqrt{1 - y^2} \), that is,
\[
\frac{d}{dx} \sin^3 x = \sqrt{1 - \sin^6 x} \\
= \cos^3 x \sqrt{(1 + 2 \sin^2 x)(1 + \sin^2 x + \sin^4 x)} \\
= \frac{3 \cos^3 x \sqrt{1 + \cos^2 x + \cos^4 x}}{(1 + 2 \cos^2 x)^3},
\]
giving the correct sign.

Similarly,
\[
\frac{d}{dx} \cos^3 x = -\sqrt{1 - \cos^6 x} \\
= -\sin^3 x \sqrt{(1 + 2 \cos^2 x)(1 + \cos^2 x + \cos^4 x)} \\
= \frac{-3 \sin^3 x \sqrt{1 + \cos^2 x + \cos^4 x}}{(1 + 2 \cos^2 x)^3},
\]

4. The general case

To generalise, we will look at the ‘rose with \( n \) petals’, where \( n \in \mathbb{N}^+ \), given in polar co-ordinates by the equation \( r^n = \sin n\theta \) (again allowing \( r \) to be negative).

![Figure 6]

Figure 6 shows one of the petals in this general case. The length \( t \) of the arc OA is given by the arc-length formula as \( \int \sqrt{1 + r^2 (\theta/\theta')^2} \, dr \). So
\[
t = \int_0^R \sqrt{1 + r^2 \frac{2n-2}{\cos^2 n\theta} \frac{d\theta}{dr}} \, dr = \int_0^R \frac{1}{\sqrt{1 - r^2 \cos^2 n\theta}} \, dr.
\]

Let us call the expression on the right-hand side \( \text{arcsin}_n R \). We will denote the inverse function of \( \text{arcsin}_n \) by \( \sin_n \), so \( R = \sin_n t \). Again after extending for an odd function we have \( R \in [-1, 1] \) and negative and positive arc lengths for \( t \). We now consider the period of the function \( \sin_n \). Let \( w_n \) be the arc length of one petal. Then
\[
\frac{1}{2} w_n = \int_0^1 \frac{1}{\sqrt{1 - r^2 \cos^2 n\theta}} \, dr.
\]

The length of the whole curve as \( \theta \) goes from 0 to \( 2\pi \) will be \( 2n w_n \) since each petal is traced out twice (allowing for positive and negative values of \( r \) when taking the root for \( n \) even). The function \( \sin_n \) can then have its domain extended by the symmetry rules
\[
\sin_n \left( \frac{w_n}{2} + t \right) = \sin_n \left( \frac{w_n}{2} - t \right)
\]
and
\[
\sin_n \left( \frac{w_n}{2} - t \right) = \sin_n \left( \frac{w_n}{2} + t \right) \text{ for } t \in \left[ 0, \frac{w_n}{2} \right]
\]
and extended again to make it periodic with period \( 2w_n \).

Using this and symmetry, we have the following equalities:
\[
\sin_n(w - t) = \sin_n(t), \\
\sin_n(w + t) = -\sin_n(t), \\
\sin_n(2w - t) = -\sin_n(t).
\]

We now define \( \cos_n(t) = \sin_n(w_n/2 - t) \), giving
\[
\cos_n(w_n + t) = -\cos_n(t), \\
\cos_n(w_n - t) = -\cos_n(t), \\
\cos_n(2w_n - t) = \cos_n(t).
\]

Since \( \text{arcsin}_n x = \int_0^x \left( 1/\sqrt{1 - r^2 \cos^2 n\theta} \right) \, dr \),
\[
\frac{d}{dx} \text{arcsin}_n x = \frac{1}{\sqrt{1 - x^2 \cos^2 n\theta}}.
\]

If \( u = \cos_n(s) \), then \( u = \sin_n(w_n/2 - s) \). So \( w_n/2 - s = \text{arcsin}_n u \), giving \( \text{arccos}_n u = w_n/2 - \text{arcsin}_n u \). Thus,
\[
\frac{d}{dx} \text{arccos}_n x = -\frac{1}{\sqrt{1 - x^2 \cos^2 n\theta}}
\]
and
\[
\text{arccos}_n u = \int_0^1 \frac{1}{\sqrt{1 - r^2 \cos^2 n\theta}} \, dr - \int_0^u \frac{1}{\sqrt{1 - r^2 \cos^2 n\theta}} \, dr
\]
\[
= \int_u^1 \frac{1}{\sqrt{1 - r^2 \cos^2 n\theta}} \, dr.
\]

Now \( \sin_n x \) and \( \cos_n x \) are periodic functions, with period \( 2w_n \), from \( \mathbb{R} \) to \([-1, 1]\). So, for \( \text{arcsin}_n t \) and \( \text{arccos}_n t \) to be functions we need to choose principal values for them. For \( \text{arcsin}_n t \) these are between \(-w_n/2\) and \( w_n/2 \) and for \( \text{arccos}_n t \) they are between \( 0 \) and \( w_n \). So
\[
\text{arcsin}_n: [-1, 1] \to \left[ -\frac{w_n}{2}, \frac{w_n}{2} \right], \\
\text{arccos}_n: [-1, 1] \to [0, \frac{w_n}{2}]\]

The graphs of \( \sin_n x \) and \( \cos_n x \) are similar to the graphs of \( \sin x \) and \( \cos x \), but as \( n \) increases the curves tend to the section of the straight line \( y = x \) from \((0, 0)\) to \((1, 1)\).

It would be natural to conjecture that \( \cos^2 t + \sin^2 t + (n - 1) \sin^2 t \cos^2 t = 1 \), but a numerical counter-example with \( n = 4 \) shows that this is not the case. For example, \( \text{arcsin}_4(0.5) = 0.50011 \), so \( \sin_4(0.50011) = 0.5 \), \( w_4/2 = 1.16359 \), \( \cos_4(0.50011) = \sin_4(0.66348) = 0.66210 \), and \( (0.66210)^2 + (0.5)^2 + 3(0.66210)^2(0.5)^2 = 1.017 \), not 1. We do have that
\[
\frac{d}{dx} \sin_n x = \sqrt{1 - \sin^2_n x}, \\
\frac{d}{dx} \cos_n x = \sqrt{1 - \cos^2_n x}
\]
We begin by stating a simple necessary and sufficient condition for a number to be a ces-number.

Lemma 1. An integer \( n \in \mathbb{N} \) is a ces-number if and only if \( n = (i + 1)(2t + i) \) for some \( i, t \in \mathbb{N} \).

Proof. An integer is a ces-number if and only if it can be expressed as \( n = 2t + (2t + 2) + (2t + 4) + \cdots + (2t + 2i) \) for some \( i, t \in \mathbb{N} \). Now, we can write the right-hand side of this expression as \( 2((i + 1)t + 1 + 2 + \cdots + i) \). Applying the well-known formula \( \sum_{j=1}^{i} j = i(i + 1)/2 \), we have \( n = (i + 1)(2t + i) \).

Let \( f(n) \) be the number of different ways in which \( n \) can be expressed as the sum of two or more consecutive even numbers. So \( f(n) \neq 0 \) if and only if \( n \) is a ces-number. Clearly, \( f(n) = 0 \) if \( n \) is odd. More precisely, we have the following lemma.

Lemma 2. Let \( n \in \mathbb{N} \). Then \( f(n) = 0 \) if and only if \( n \) is odd or \( n = 2r \) for some \( r \in \mathbb{N} \).

Proof. We first show that, if \( n \) is odd or \( n = 2r \), then \( f(n) = 0 \). Equivalently, we show that, if \( f(n) \neq 0 \), then \( n \neq 2r \) and \( n \) is even. From lemma 1, we know that \( f(n) \neq 0 \) if and only if \( n = (i + 1)(2t + i) \) for some \( i, t \in \mathbb{N} \). If \( i \) is odd, then \( i + 1 \) is even and \( 2t + i \) is odd, so \( n \) is the product of an odd number (\( > 1 \)) and an even number. The same is true if \( i \) is even. Thus, since \( n \) is a multiple of an even number, it too must be even. But, since \( n \) is also a multiple of an odd number, it cannot be of the form \( n = 2r \) for some \( r \in \mathbb{N} \).

Next we show that, if \( f(n) = 0 \), then \( n \) is odd or \( n = 2r \). In fact, we show that, if \( n = ab \) where \( a > 1 \) is odd and \( b \) is even, then \( f(n) \neq 0 \) (which is clearly equivalent). There are two cases to consider. Firstly, assume that \( a > b \). Then take \( i = b - 1 \) and \( t = (a - b + 1)/2 \). It is easy to see that \( i \) and \( t \) are positive integers and that \( n = (i + 1)(2t + i) \), so that \( f(n) \neq 0 \) by lemma 1. Similarly, if \( a < b \), then take \( i = a - 1 \) and \( t = (b - a + 1)/2 \). Again \( i \) and \( t \) are positive integers and \( n = (i + 1)(2t + i) \), so that \( f(n) = 0 \).

For any positive integer \( n \in \mathbb{N} \), we define \( S(n) \) to be the number of different ways that \( n \) can be expressed as the difference of two squares, \( n = x^2 - y^2 \), with both \( x \) and \( y \) odd integers. Since \( x^2 - y^2 = (x - y)(x + y) \), we will be able to use the above result along with the following lemma to obtain a formula for \( f(n) \).

Lemma 3. Let \( n \in \mathbb{N} \) be a ces-number. Then \( f(n) = S(4n) - 1 \).

Proof. We first show that \( f(n) \leq S(4n) - 1 \). Take any representation of \( n \) as a sum of consecutive even numbers, say \( n = 2t + (2t + 2) + (2t + 4) + \cdots + (2t + 2i) \), where \( i, t \in \mathbb{N} \). Then, by lemma 1, \( n = (i + 1)(2t + i) \),
Thus, from the expression $4n = x^2 - y^2$ for some $n \in \mathbb{N}$ with $x$ odd.

Now, since $t$ is a positive integer, it follows that $t^2 - t + \frac{1}{4} + n = x^2/4$ for some $x \in \mathbb{N}$ with $x$ odd. It then follows that $(t - \frac{1}{2})^2 = x^2/4 - n$. Again, since $t$ is an integer, it must be the case that $x^2/4 - n = y^2/4$ for some $y \in \mathbb{N}$ with $y$ odd (namely $y = 2t - 1$), and so

$$4n = x^2 - y^2.$$  

Since $x$ and $y$ are uniquely determined by $i$ and $t$, we can conclude that $f(n) \leq S(4n) - 1$. The term $-1$ is due to the fact that there are no $x$ and $y$ corresponding to $4n = (n+1)^2 - (n-1)^2$: this would imply that $x = 2t+1$ and $y = 2t-1$; since $4n = x^2 - y^2$, we would obtain that $n = 2t$ and this implies that $t = 0$, a contradiction.

We now show that $f(n) \geq S(4n) - 1$. Suppose that $4n = x^2 - y^2$ with $x, y$ odd and $(x-y)/2 > 1$, i.e. any of the $S(4n)$ ways of writing $4n = x^2 - y^2$ except $4n = (n+1)^2 - (n-1)^2$. Note that $m^2 = 1 + 3 + 5 + \cdots + (2m-1)$ for any $m \in \mathbb{N}$, so

$$x^2 - y^2 = 1 + 3 + 5 + \cdots + (2y-1) + (2y+1) + \cdots + (2x-1) = (y+1) + (y+3) + \cdots + (x-1).$$

Taking these terms in consecutive pairs (which is possible since $x$ and $y$ are odd) and dividing by 4, we obtain

$$n = (y+1) + (y+3) + \cdots + (x-1).$$

Thus, from the expression $4n = x^2 - y^2$, we have obtained a representation of $n$ as a sum of consecutive even numbers since $x$ and $y$ are odd. We can also see that, for different $x, y$ these representations are non-trivial and different since they have length $(x-y)/2 > 1$. 

It now remains to determine $S(4n)$ for any $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Then the ways of writing $m$ as $x^2 - y^2$, where $x, y \in \mathbb{N}$, are in one-to-one correspondence with the ways of factoring $m = ab$ with $a$ and $b$ either both even or both odd since, if $m = x^2 - y^2$, then $m = (x-y)(x+y)$, and, if $m = ab$ (with $a$ and $b$ both odd or both even), then

$$m = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2.$$

### Theorem 1

Let $n \in \mathbb{N}$ be a cs-number and suppose that $n = 2^{a_1}p_2^{b_1}\cdots p_r^{b_r}$ for some odd primes $p_2, \ldots, p_r$ and some integers $a_i \geq 1$. Then

$$f(n) = (a_2 + 1)(a_3 + 1) \cdots (a_r + 1) - 1.$$

**Proof.** From the correspondence given above (and lemma 3), $f(n) + 1$ is the number of ways of writing $4n$ as the product $ab$ of two even numbers, where one factor is 2 times an odd number. In other words, $f(n) + 1$ is the number of ways of writing $n$ as a product $cd$, where $c$ is odd and $d$ is even. The result then follows easily.

For example, the result tells us that $f(12) = f(2^3) = 1$; the unique expression of 12 as the sum of consecutive even numbers is $12 = 2 + 4 + 6$. Note that it follows that, if $n$ is even, then $f(n) = f(2n)$ for every $r \in \mathbb{N}$.

**Corollary 1.** Let $n \in \mathbb{N}$ be a cs-number. Then $f(n) = 1$ if and only if $n = 2^p$ for some odd prime $p$ and some $r \in \mathbb{N}$.

**Proof.** This follows directly from theorem 1.

In a similar way, we can note that $f(n) = 2$ if and only if $n = 2^p 3^r$ for some odd prime $p$ and $r \in \mathbb{N}$.

### 3. The number of sums of consecutive numbers

Recall that a positive integer $n$ is a cs-number if it can be expressed as the sum of two or more consecutive numbers. Let $F(n)$ be the number of ways of representing $n$ in this way. It is easy to see that $F(n) = f(2n)$. Hence, we have the following result, the first part of which was shown in references 1 and 2.

**Corollary 2.** Let $n \in \mathbb{N}$. Then $n$ is a cs-number if and only if it is not a power of 2. Furthermore, if $n = 2^{a_1}p_2^{b_1}\cdots p_r^{b_r}$ for some odd primes $p_2, \ldots, p_r$ with $a_1$ possibly zero, then

$$F(n) = (a_2 + 1)(a_3 + 1) \cdots (a_r + 1) - 1.$$

**Proof.** The first part follows from the observation that $F(n) = f(2n)$ and lemma 2. The second part follows again since $F(n) = f(2n)$ and from theorem 1.

### References


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Complex numbers are a powerful tool used in many areas of mathematics, science and engineering, and Carl Friedrich Gauss, often referred to as the ‘Prince of Mathematicians’, was one of the first to give a coherent account of these. One popular application of complex numbers is in finding sums of finite and infinite series involving sines and cosines, and often based on geometric series. For example, starting with the series

\[
C = \frac{1}{2} \cos \theta + \frac{1}{2^2} \cos 2\theta + \frac{1}{2^3} \cos 3\theta + \cdots,
\]
defining the corresponding series with sines

\[
S = \frac{1}{2} \sin \theta + \frac{1}{2^2} \sin 2\theta + \frac{1}{2^3} \sin 3\theta + \cdots
\]
and then combining (1) and (2) using the imaginary unit \(i = \sqrt{-1}\), we have

\[
C + iS = \frac{1}{2} e^{i\theta} + \frac{1}{2^2} e^{2i\theta} + \frac{1}{2^3} e^{3i\theta} + \cdots = \frac{1}{1 - \frac{1}{2} e^{i\theta}}.
\]

We have

\[
C + iS = \frac{\frac{1}{2} e^{i\theta} - 1}{\frac{1}{2} e^{i\theta} - 1} = \frac{\frac{1}{2} e^{i\theta} - 1}{1 - \frac{1}{2} e^{i\theta}} - \frac{\frac{1}{2} e^{i\theta} - 1}{1 - \frac{1}{2} e^{-i\theta}} = \frac{e^{i\theta}}{1 + \frac{1}{4} - \frac{1}{4} (e^{i\theta} + e^{-i\theta})} = \frac{1}{2} (\cos \theta + i \sin \theta) - \frac{1}{4} \sin \theta
\]

Taking the real part of (7), we have

\[
C = \frac{1}{2} \cos \theta + \frac{1}{2^2} \cos 2\theta + \frac{1}{2^3} \cos 3\theta + \cdots = \frac{\cos \theta - 1}{5 - 4 \cos \theta}
\]
and taking the imaginary part we obtain the sum of the sine series (2).

On close inspection of (8), we notice that, if \(\theta = \pi/3\), then \(\cos \theta = -\frac{1}{2}\) and the right-hand side vanishes, i.e.

\[
\frac{1}{2} \cos \frac{\pi}{3} + \frac{1}{2^2} \cos \frac{2\pi}{3} + \frac{1}{2^3} \cos \frac{3\pi}{3} + \cdots = 0,
\]

which I found surprising, and felt needed further explanation. Since \(\cos \pi/3 = \frac{1}{2}\), \(\cos 2\pi/3 = -\frac{1}{2}\), \(\cos 3\pi/3 = -1\), \(\cos 4\pi/3 = -\frac{1}{2}\), \(\cos 5\pi/3 = -\frac{1}{2}\) and \(\cos 6\pi/3 = 1\), with all further terms just repeats of these through periodicity, the left-hand side of (9) is

\[
\frac{1}{2} \left( \frac{1}{2} \right) + \frac{1}{2^2} \left( \frac{1}{2^2} \right) + \frac{1}{2^3} \left( \frac{1}{2^3} \right) + \cdots = \frac{1}{1 - \frac{1}{2}} \left( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \frac{1}{2} \left( \frac{1}{2} \right) + \frac{1}{2^2} \left( \frac{1}{2^2} \right) + \frac{1}{2^3} \left( \frac{1}{2^3} \right) + \cdots = \frac{1}{2} \left( \frac{1}{2} \right) + \frac{1}{2^2} \left( \frac{1}{2^2} \right) + \frac{1}{2^3} \left( \frac{1}{2^3} \right) + \cdots
\]

which gives the results above.
We leave readers to show that the corresponding sum is

\[
\frac{1}{2} - \frac{1}{4} - \frac{1}{16} + \frac{1}{32} + \frac{1}{32} \times \frac{1/2}{1 - 1/2^6} = 0 \times \frac{32}{63} = 0,
\]

confirming the result in (9).

I then became intrigued as to what might happen for

\[
C = a \cos \theta + a^2 \cos 2\theta + a^3 \cos 3\theta + \cdots, \tag{10}
\]

an obvious generalisation of (1), where \(a\) is real with \(|a| < 1\).

We leave readers to show that the corresponding sum is

\[
C = \frac{a \cos \theta - a^2}{1 + a^2 - 2a \cos \theta}, \tag{11}
\]

and this is zero when \(\cos \theta = a\), so, from (10) and (11),

\[
\cos \theta \cos \theta + \cos^2 \theta \cos 2\theta + \cos^3 \theta \cos 3\theta + \cdots = 0,
\]

a fascinating result I am sure you agree. For example,

\[
\frac{1}{3} \cos \theta + \frac{1}{3^2} \cos 2\theta + \frac{1}{3^3} \cos 3\theta + \cdots = 0
\]

when \(\cos \theta = \frac{1}{3}\). But there is more …

A more complicated series than (1) is

\[
C = 1 \times \frac{1}{2} \cos \theta + 2 \times \frac{1}{2^2} \cos 2\theta + 3 \times \frac{1}{2^3} \cos 3\theta + \cdots,
\]

where the individual terms have been multiplied by the natural numbers. Repeating the steps leading up to (3) by forming \(C + iS\) using the corresponding sine series \(S\), we obtain

\[
C + iS = 1 \times \frac{1}{2} e^{i\theta} + 2 \times \frac{1}{2^2} (e^{i\theta})^2 + 3 \times \frac{1}{2^3} (e^{i\theta})^3 + \cdots, \tag{12}
\]

which is an arithmetic-geometric series. The formula for the sum in (12) is readily derived by differentiating (6) with respect to \(r\) to give

\[
\alpha + 2\alpha r + 3\alpha r^2 + \cdots = \frac{\alpha}{(1 - r)^2},
\]

which is also convergent when \(|r| < 1\), and substituting \(\alpha = r = \frac{1}{2} e^{i\theta}\), so that

\[
C + iS = \frac{\frac{1}{2} e^{i\theta}}{(1 - \frac{1}{2} e^{i\theta})^2}. \tag{13}
\]

Multiplying the numerator and denominator of (13) by the complex conjugate

\[
(1 - \frac{1}{2} e^{i\theta})^2 = (1 - \frac{1}{2} e^{-i\theta})^2,
\]

we have

\[
C + iS = \frac{\frac{1}{2} e^{i\theta}}{(1 - \frac{1}{2} e^{i\theta})^2} \frac{(1 - \frac{1}{2} e^{-i\theta})^2}{(1 - \frac{1}{2} e^{-i\theta})^2} = \frac{\frac{1}{2} e^{i\theta}(1 + \frac{1}{2} e^{-2i\theta} - e^{-i\theta})}{((1 - \frac{1}{2} e^{i\theta})(1 - \frac{1}{2} e^{-i\theta}))^2} = \frac{\frac{1}{2} e^{i\theta} + \frac{1}{2} e^{-i\theta} - 1}{(\frac{1}{4} - \cos \theta)^2} \times 8e^{i\theta} + 2e^{-i\theta} - 8
\]

\[
= \frac{10 \cos \theta - 8 + 6i \sin \theta}{(5 - 4 \cos \theta)^2}, \tag{14}
\]

and, taking the real part of (14),

\[
C = 1 \times \frac{1}{2} \cos \theta + 2 \times \frac{1}{2^2} \cos 2\theta + 3 \times \frac{1}{2^3} \cos 3\theta + \cdots
\]

\[
= \frac{10 \cos \theta - 8}{(5 - 4 \cos \theta)^2}. \tag{15}
\]

This time, the sum of the series in (15) is zero when \(\cos \theta = \frac{5}{8}\), which does not seem particularly special or interesting. After a moment’s thought, however, we recall that \(\theta\) is the smaller acute angle in a \((3, 4, 5)\) Pythagorean triangle. Is this a coincidence?

Quickly turning to another case, say

\[
C = 1 \times \frac{1}{3} \cos \theta + 2 \times \frac{1}{3^2} \cos 2\theta + 3 \times \frac{1}{3^3} \cos 3\theta + \cdots
\]

(saving the general case to maintain the suspense!), readers may like to show that the algebra now gives

\[
C = 1 \times \frac{1}{3} \cos \theta + 2 \times \frac{1}{3^2} \cos 2\theta + 3 \times \frac{1}{3^3} \cos 3\theta + \cdots
\]

\[
= \frac{30 \cos \theta - 18}{(10 - 6 \cos \theta)^2}
\]

which is zero when \(\cos \theta = \frac{3}{5}\), i.e. when \(\theta\) is now the larger acute angle in a \((3, 4, 5)\) Pythagorean triangle, so nothing really new has been discovered.

Pressing on with the next case, readers will find that

\[
C = 1 \times \frac{1}{4} \cos \theta + 2 \times \frac{1}{4^2} \cos 2\theta + 3 \times \frac{1}{4^3} \cos 3\theta + \cdots
\]

\[
= \frac{68 \cos \theta - 32}{(17 - 8 \cos \theta)^2}
\]

so that the sum is zero when \(\cos \theta = \frac{8}{17}\), i.e. when \(\theta\) is the larger acute angle in an \((8, 15, 17)\) Pythagorean triangle.

For the general case

\[
C = 1 \times a \cos \theta + 2 \times a^2 \cos 2\theta + 3 \times a^3 \cos 3\theta + \cdots
\]

the sum can be determined in the same way as (15) by replacing \(\frac{1}{2}\) by \(a < 1\) in the first line of (14). After some simplification, the corresponding sum is

\[
C = 1 \times a \cos \theta + 2 \times a^2 \cos 2\theta + 3 \times a^3 \cos 3\theta + \cdots
\]

\[
= \frac{(a + a^3) \cos \theta - 2a^2}{(1 - 2a \cos \theta + a^3)^2} \tag{16}
\]
and this is zero when
\[ \cos \theta = \frac{2a^2}{a + a^3} = \frac{2a}{1 + a^2}. \]
Setting \( a = n/m \) for positive integers \( m, n \) with \( m > n \), we have
\[ \cos \theta = \frac{2n/m}{1 + n^2/m^2} = \frac{2mn}{m^2 + n^2}. \]
so that \( \theta \) is again one of the angles in a Pythagorean triangle, this time with sides \( m^2 - n^2, 2mn \) and \( m^2 + n^2 \) since \( (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 \). (Interestingly, the denominator in (16) can only vanish when \( \cos \theta = (1 + a^2)/2a \), which happens to be the reciprocal of the value above, and of course has no real solutions for \( \theta \).

For example, when \( m = 3 \) and \( n = 2 \), \( a = n/m = \frac{2}{3} \) and (16) becomes
\[ 1 \times \frac{2}{3} \cos \theta + 2 \times \left(\frac{2}{3}\right)^2 \cos 2\theta + \frac{2}{3} \times \left(\frac{2}{3}\right)^3 \cos 3\theta + \cdots = 0 \]
when
\[ \cos \theta = \frac{2mn}{m^2 + n^2} = \frac{12}{13}, \]
so that \( \theta \) is the smaller acute angle in a \( (5, 12, 13) \) Pythagorean triangle.
Now I wonder what Pythagoras would have thought of that?

The author lectures in mathematics at Reading University.

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**Mathematics in the Classroom**

The binomial and negative binomial distributions

Consider an experiment with two possible outcomes, not necessarily equally likely, that are labelled generically ‘success’ and ‘failure’. In such circumstances, the binomial distribution allows us to calculate the probability of obtaining exactly \( x \) successes and \( y \) failures in \( n \) trials as follows:

\[ \Pr(x \text{ successes}) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n, \]

where \( 0 < p < 1 \) is the probability of a success and \( q = 1 - p \) is the probability of a failure. The mean and variance of this distribution are given by \( \mu = np \) and \( \sigma^2 = npq \) respectively.

To illustrate, let the experiment consist of cutting a pack of playing cards with a success being an ace with probability \( p = \frac{1}{13} \) and failure being any other card with probability \( q = \frac{12}{13} \). Therefore, the probability of obtaining, say, exactly two aces and four non-aces in six trials is given by

\[ 6 \binom{1}{2} \left(\frac{1}{13}\right)^2 \left(\frac{12}{13}\right)^{12} = 0.0819. \]

Suppose now that we wish to calculate the probability that the \( n \)th trial produces the \( x \)th success. (Note that this is subtly different from \( n \) trials producing \( x \) successes.)

For the \( n \)th trial to produce the \( x \)th success, the first \( n - 1 \) trials must have produced exactly \( x - 1 \) successes and therefore \( (n - 1) - (x - 1) = n - x \) failures and the \( n \)th trial must produce a success. The probability of this sequence of events is given by

\[ \Pr(n\text{th trial gives } x \text{th success}) = \binom{n-1}{x-1} p^{x-1} q^{n-x} \]

or

\[ \binom{n-1}{x-1} p^{x-1} q^{n-x}, \quad n = x, x+1, \ldots. \]

The above equation represents the negative binomial or Pascal distribution, which appears in the literature in a variety of equivalent guises, viz.

\[ \binom{x+y-1}{x-1} p^x q^y \quad \text{or} \quad \binom{n-1}{x-1} p^x q^y. \]

The mean and variance of this distribution are given by

\[ \mu = \frac{xq}{p} = \frac{x(1-p)}{p} \]

and

\[ \sigma^2 = \frac{xq}{p^2} = \frac{x(1-p)}{p^2} \]

respectively.

Returning to our card-cutting experiment for illustration, the probability of the seventh cut producing the third ace is thus

\[ \binom{7-1}{3-1} p^3 q^4 = 0.0063. \]

The interested reader can easily confirm that any version of the negative binomial above will duplicate this result.

From the foregoing, we can see how the binomial and negative binomial distributions differ from one another. A binomial random variable is a count of the number, \( x \), of successes in a predetermined number, \( n \), of trials, i.e. the
number, \( n \), of trials is fixed and the number, \( x \), of successes is random. Conversely, a negative binomial random variable is a count of the number, \( n \), of trials required to obtain a predetermined number, \( x \), of successes, i.e. the number of successes, \( x \), is fixed and the number of trials, \( n \), is random. In this sense, the negative binomial can be considered the opposite, or negative, of the binomial.

**Fitting a negative binomial distribution: an example from soccer**

During the 2002/03 soccer season, Kilmarnock F.C. scored 47 goals over 38 league games and finished fourth in the Scottish Premier League. The actual distribution of these goals is shown in table 1 from which it can be calculated that the mean number of goals per game was 1.24 with a variance of 1.39.

<table>
<thead>
<tr>
<th>Number of goals (( y ))</th>
<th>Observed frequency of ( y ) goals</th>
<th>Theoretical frequency of ( y ) goals</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total games</td>
<td>38</td>
<td>38</td>
</tr>
</tbody>
</table>

Since football teams vary their players and tactics and play against opposition of varying calibre on grounds of varying quality, it is unrealistic to assume that the probability of a goal, and hence the expected number of goals for a particular team, will remain constant from one game to another. In such circumstances, as Pollard has suggested (see reference 1), the appropriate statistical distribution to fit to goal-scoring data may be the negative binomial distribution rather than the more popular and easily computed Poisson distribution.

To fit a negative binomial distribution to the above data, we make use of the following:

\[ \mu = \frac{x(1-p)}{p} \quad \text{and} \quad \sigma^2 = \frac{x(1-p)}{p^2} \]

from which

\[ p = \frac{x}{\mu - x} \quad \text{and} \quad q = \frac{\mu}{\mu + x} \]

and

\[ x = \frac{\mu^2}{\sigma^2 - \mu} \quad \text{and} \quad p = \frac{\mu}{\sigma^2}. \]

Thus, by substitution,

\[ \frac{(x + y - 1)}{y} p^x q^y = \left( \frac{x + y - 1}{y} \right) \left( \frac{x}{\mu + x} \right)^y \left( \frac{\mu}{\mu + x} \right)^y, \]

where \( y = 0, 1, 2, 3, \ldots, x > 0 \) and \( 0 < p < 1 \).

As indicated above, \( \mu = 1.24 \) and \( \sigma^2 = 1.39 \), which allows us to calculate that

\[ x = \frac{1.24^2}{1.39 - 1.24} = 10.25 \]

and

\[ p = \frac{1.24}{1.39} = 0.89 \]

so that \( q = 0.11 \). Since \( x \) is an integer by definition, it should be rounded off to \( x = 10 \). If we now define \( y \) as the number of goals scored per game, then the probability of scoring, say, \( y = 4 \) goals is

\[ \left( \frac{10 + 4 - 1}{4} \right) (0.89)^4 (0.11)^4 = 0.03264. \]

Thus, the theoretical or expected frequency of four goals per game is \( 38 \times 0.03264 = 1.24 \), or 1 rounded to the nearest whole number. The theoretical frequency of \( y = 0, 1, 2, \ldots, 6 \) goals is shown in table 1.

We may now apply the familiar \( \chi^2 \) goodness-of-fit test in the usual way. Pooling the theoretical frequency of 3, 4, 5 and 6 goals to ensure that all such frequencies equal 5 or more, a \( \chi^2 \) of 3.57 indicates that the negative binomial distribution provides an acceptable fit to this data.

**Reference**


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**John C. B. Cooper**

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**Prove the cosine formula at a glance**

Use the diagram to prove the cosine formula

\[ c^2 = a^2 + b^2 - 2ab \cos C. \]

GUIDO LASTERS
Tienen, Belgium.
Dear Editor,

I read with interest Seyamack Jafari’s ‘A polygon spiral’ in Mathematical Spectrum, Volume 35, Number 3. From figure 1 we have

\[
\sin \theta = \frac{1}{x_0}, \quad \cos \theta = \frac{1}{x_1},
\]

\[
\sin^2 \theta + \cos^2 \theta = 1, \quad \frac{1}{x_0^2} + \frac{1}{x_1^2} = 1, \tag{1}
\]

\[
a^2 = x_0^2 + x_1^2 = x_0 \times x_1,
\]

\[
\sin \phi = \frac{1}{a}, \quad \cos \phi = \frac{1}{x_2},
\]

therefore,

\[
\frac{1}{a^2} + \frac{1}{x_2^2} = 1
\]

and

\[
b^2 = a^2 + x_2^2 = a^2 + x_0 \times x_1 = x_0^2 + x_1^2.
\]

Also, by (1),

\[
b^2 = a^2 + x_2^2 = x_0^2 + x_1 + x_2^2.
\]

Continue this way for \(c, d\) etc.

The question arises: can we find rational numbers \(x_0^2, x_1^2, x_2^2, \ldots\) that satisfy

\[
x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = x_0^2 \times x_1^2 \times x_2^2 \times \cdots \times x_n^2.
\]

Put \(\theta = 30^\circ\). Then

\[
x_0 = \frac{1}{\sin 30^\circ} = 2, \quad x_1 = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}
\]

and

\[
2^2 + \left(\frac{2}{\sqrt{3}}\right)^2 = 2^2 \times \left(\frac{2}{\sqrt{3}}\right)^2,
\]

that is,

\[
4 + \frac{4}{3} = 4 \times \frac{4}{3}.
\]

Further,

\[
\sin^2 \phi = \frac{1}{a^2} = \frac{1}{x_0^2 x_1^2} = \frac{3}{16}, \quad \cos^2 \phi = \frac{1}{x_2^2},
\]

Hence,

\[
\frac{3}{16} + \frac{1}{x_2^2} = 1,
\]

whence, \(x_2^2 = \frac{16}{13}\). Then

\[
x_0^2 + x_1^2 + x_2^2 = 4 + \frac{4}{3} + \frac{16}{13} = \frac{256}{39}
\]

and

\[
x_0^2 \times x_1^2 \times x_2^2 = 4 \times \frac{4}{3} \times \frac{16}{13} = \frac{256}{39}.
\]

Continuing,

\[
\sin \alpha = \frac{1}{b}, \quad \cos \alpha = \frac{1}{x_3}, \quad \frac{1}{b^2} + \frac{1}{x_3^2} = 1,
\]

\[
c^2 = b^2 + x_3^2 = b^2 x_3^2 = x_0^2 \times x_1^2 \times x_2^2 \times x_3^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{(by 2)}
\]

and

\[
\sin^2 \alpha = \frac{1}{b^2} = \frac{1}{x_0^2 x_1^2 x_2^2} = \frac{1}{4 \times \frac{4}{3} \times \frac{16}{13}} = \frac{39}{256},
\]

\[
\cos^2 \alpha = \frac{1}{x_3^2},
\]

Hence,

\[
\frac{39}{256} + \frac{1}{x_3^2} = 1,
\]

whence, \(x_3^2 = \frac{256}{217}\). Thus,

\[
x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 \times x_1^2 \times x_2^2 \times x_3^2 = \frac{65536}{8465}
\]

It will be found that, for \(n > 0\),

\[
x_n^2 = \frac{a_n}{b_n}, \quad \text{where} \quad a_n = 2^n \quad \text{and} \quad b_n = a_n - \prod_{i=0}^{n-1} b_i,
\]

with \(b_0 = 1\), and the sum (or product) of the \(x_i^2\) up to and including \(x_n^2\) is given by

\[
\frac{c_n}{d_n}, \quad \text{where} \quad c_n = 2^{n+1} \quad \text{and} \quad d_n = \prod_{i=1}^{n} b_i,
\]

all of which increase quite rapidly.

Other values for \(\cos\) and \(\sin\) will give other solutions.

Yours sincerely,

Bob Bertuello
(12 Pinewood Road, Midsomer Norton, Bath BA3 2RG, UK.)
Dear Editor,

π day, Sunday, March 14, 1:59pm

The number is π, 3.1415926535 . . . . It’s the number you get when you divide the circumference of a circle by its diameter, and it can’t be expressed as a fraction; it goes on forever without repetition. In this era when maths and mathematicians have become sexy again, come to San Francisco’s Exploratorium and gather around the π shrine to perform π-related rites and eat ritual food, pie, in honour of this special number. And come to the Exploratorium and attend the US premier of Porridge Pulleys and Pi: Two Mathematical Journeys, directed by George Paul Csicsery, presented in conjunction with the Mathematical Sciences Research Institute, Berkeley. It’s all part of π day celebrations, which culminate, appropriately enough, at 1:59pm on March 14. That’s the third month, the fourteenth day, at 1:59pm, corresponding to the approximation 3.14159 to π.

In addition to the film and other activities, at 1:59pm celebrants are invited to circumambulate the π shrine approximately 3.14 times. The π shrine is a small brass plate engraved with π to a hundred digits located on the floor on the mezzanine level of the Exploratorium, the prototype for hands-on science museums around the world. Add beads to a ritual π string, where each bead color designates a value for π to over 1600 digits, and growing. Music based on the number π provides the ambience.

The film Porridge Pulleys and Pi: Two Mathematical Journeys, features Vaughan Jones, one of the world’s foremost knot theorists, and Hendrik Lenstra, a number theorist, as well as everything from genomics, music, and elliptic curves to Homer and the history of π. Meet Vaughan Jones and other real live mathematicians from MSRI following the screening.

Most important of all, people eat pie. Lemon meringue, apple, peach, whatever. The public is invited to attend π day. It would be nice also to bring along a pie to share. Even though the publication of this letter misses π day, we thought readers would be interested to know what they have missed. Maybe next year! — Ed.[

Dear Editor,

‘Squarps’ and the lead–end digit trade

Consider the process of taking a positive integer whose leading and ending digits are distinct and none of whose digits are zeros, and exchanging the leading and ending digits. (We will only consider numbers without zeros for digits as this will eliminate any ambiguity that arises when zeros are dropped in the switching procedure.) Let’s call this process the ‘lead–end digit trade’. For example, 12 345 after the lead–end digit trade becomes 52 341.

Now let’s look at the number 1156, a perfect square containing no zeros for digits. The number 6151 obtained by exchanging its leading and ending digits turns out to be a prime. Let’s refer to perfect squares which give a prime number like this as squarps for short. I had the idea for this letter after reading Chapter 80 on ‘Parasite Numbers’ in reference 1, in particular the last question in the ‘Furthering Exploring’ section. However, the chapter mentioned deals with a different problem entirely.

If we use a computer to search for squarps we will find a lot of them. I used the software package PARI-GP, which is freely available from the Internet (see reference 2). The first few squarps are 16, 196, 361, 784, 1156, 1444, 3136, 3364, 3481, 3721, 7225, 7396, 11 236, 12 769, 15 376.

Since squarps are relatively abundant, what would a plot of these numbers look like? Figure 1 shows the points (n, n2) such that n2 is a squarp for 0 < n < 1000. This graph seems to suggest that squarps come in ‘clumps’ with large gaps between them. What would a plot of squarps for 0 < n < 100000 tell us?

![Figure 1. Distribution of squarps.](image)

This suggests some generalizations and some questions.

- Squarcs. Find perfect squares that become perfect cubes after performing the lead–end digit trade. I haven’t found any squarcs.
- Squarts. Find perfect squares that become triangular numbers after performing the lead–end digit trade. The first squart is 5 239 521 = 22892 since 1 239 525 = 1574 × (1574 + 1)/2. Three others are 69 172 489, 62 154 977 481 and 88 844 532 624.

References


Yours sincerely,

JASON EARLS
(516 W. Oklahoma, Blackwell, OK 74631, USA.)
Problems

36.9 Let \( a, b, c, d \) be positive real numbers such that \( a + b \geq c + d \) and \( ab \leq cd \). Prove that \( a^m + b^m \geq c^m + d^m \) for all integers \( m \).

(Submitted by Hassan Shah Ali, Tehran, Iran)

36.10 When \( n \) people each throw a pair of dice, what is the probability that they all get the same total?

(Submitted by Anand Kumar, Patna, India)

36.11 The point \( P \) moves in a plane so that the tangents through \( P \) to a given ellipse are at right angles. Determine the locus of \( P \). If these tangents touch the ellipse at \( M, N \), find the envelope of \( MN \).

(Submitted by Seyamack Jafari, Khozestan, Iran)

36.12 What is
\[
\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \ ?
\]

(Submitted by Zhang Yun, The First Middle School of Jinchang City, China)

Solutions to Problems in Volume 36 Number 1

36.1 Let \( A_1 A_2 \ldots A_n \) be a regular \( n \)-sided polygon. Show that \( (PA_1)^2 + (PA_2)^2 + \cdots + (PA_n)^2 \) is the same for all points \( P \) on the circumscribed circle of the polygon.

Solution by J. Craig, who proposed the problem

Let

\[
A_k = \left( r \cos \frac{2\pi (k - 1)}{n}, r \sin \frac{2\pi (k - 1)}{n} \right)
\]

for \( k = 1, \ldots, n \), and

\[
P = (r \cos \theta, r \sin \theta).
\]

Then

\[
\sum_{k=1}^{n} (PA_k)^2 = r^2 \sum_{k=1}^{n} \left[ \left( \cos \frac{2\pi (k - 1)}{n} - \cos \theta \right)^2 + \left( \sin \frac{2\pi (k - 1)}{n} - \sin \theta \right)^2 \right]
\]

\begin{align*}
&= 2nr^2 - 2r^2 \sum_{k=1}^{n} \left( \cos \frac{2\pi (k - 1)}{n} \cos \theta + \sin \frac{2\pi (k - 1)}{n} \sin \theta \right) \\
&= 2nr^2 - 2r^2 \sum_{k=1}^{n} \cos \left( \frac{2\pi (k - 1)}{n} - \theta \right) \\
&= 2nr^2 - 2r^2 \text{Re} \left\{ \sum_{k=1}^{n} e^{\frac{2\pi (k - 1)}{n} - \theta} \right\} \\
&= 2nr^2 - 2r^2 \text{Re} \left\{ e^{-i\theta} \left( 1 - e^{\frac{2\pi i}{n}} \right) \right\} \\
&= 2nr^2.
\end{align*}

J. Craig points out that the same holds true when \( P \) lies on any circle concentric to the circumscribed circle of the polygon. If the circle on which \( P \) lies has radius \( R \), then

\[
\sum_{k=1}^{n} (PA_k)^2 = n(r^2 + R^2).
\]

A similar point is made by S. Hayes.

John MacNeill, one of our editors, points out that a more general result is true, namely that, in 2- or 3-dimensional space, if \( A_1, A_2, \ldots, A_n \) are points equidistant from a point \( O \) the sum of whose position vectors relative to \( O \) is zero, then \( PA_1^2 + \cdots + PA_n^2 \) is constant. To see this, we denote the position vector of \( A_k \) relative to \( O \) by \( a_k \) for \( k = 1, \ldots, n \) and the position vector of \( P \) by \( r \). Put \( |a_1| = \cdots = |a_n| = h \) and \( |r| = \ell \). Then

\[
PA_k^2 = (r - a_k) \cdot (r - a_k) = |r|^2 - 2r \cdot a_k + |a_k|^2,
\]

so

\[
\sum_{k=1}^{n} (PA_k)^2 = n\ell^2 - 2r \cdot (a_1 + \cdots + a_n) + nh^2 = n(h^2 + \ell^2).
\]

36.2 A 3×3 multiplication magic square is defined as a square array of natural numbers in which the sums of the elements in each row, each column and on each of the two diagonals are the same, \( s \) say. Express the element in the centre in terms of \( s \) and show how to obtain different 3×3 magic squares with a given central element. Define a 3×3 multiplication magic square by replacing addition by multiplication. What are the corresponding results for these?
Solution by Emanuel Emanouilidis, who proposed the problem

Denote the central element by \( x \). If we add the two diagonals and the middle row and column, we see that

\[
4s = 3s + 3x,
\]

so \( x = s/3 \). All squares of the form

\[
\begin{array}{ccc}
   x+a & x-a-b & x+b \\
   x+b-a & x & x+a-b \\
   x-b & x+a+b & x-a
\end{array}
\]

where \( a, b \) are natural numbers such that \( a + b < x \), are magic squares with central element \( x \).

For a multiplication magic square, if we multiply instead of add, we obtain \( x = p^{1/3} \), where \( p \) is the product of the entries in a row, column or diagonal. All squares of the form

\[
\begin{array}{ccc}
   xa & \frac{xa}{ab} & xb \\
   \frac{xb}{a} & x & \frac{xb}{b} \\
   \frac{x}{b} & \frac{xab}{x} & \frac{x}{a}
\end{array}
\]

will be multiplication magic squares with central element \( x \), where \( a, b \) are positive divisors of \( x \).

36.3 The positive real numbers \( a_1, \ldots, a_n \) are such that \( a_1 + a_2 + \cdots + a_n = n \). Show that \( a_1^{1/2} + a_2^{1/2} + \cdots + a_n^{1/2} \geq n \).

Solution by Milton Chowdhury, who proposed the problem

Since

\[
1 + \delta \leq \left(1 + \frac{\delta}{2}\right)^2,
\]

when \( \delta > -1 \),

\[
(1 + \delta)^{1/2} \leq 1 + \left(\frac{\delta}{2}\right).
\]

Hence,

\[
(1 + \delta)^{3/2} = \frac{(1 + \delta)^2}{(1 + \delta)^{1/2}} \geq \frac{(1 + \delta)^2}{1 + \delta/2} \geq 1 + \frac{3\delta}{2}.
\]

Let \( \delta = a_i - 1 \). Then

\[
a_i^{3/2} \geq \frac{3a_i}{2} - \frac{1}{2},
\]

so

\[
\sum_{i=1}^{n} a_i^{3/2} \geq \frac{3n}{2} - \frac{n}{2} = n.
\]

36.4 Let \( s_n = n! - n^n/e^{n-1} \) for \( n = 1, 2, 3, \ldots \). Prove that the sequence \( (s_n) \) is strictly increasing and unbounded.

Solution by Hassan Shah Ali, who proposed the problem

We have

\[
s_{n+1} = (n+1)! - \frac{(n+1)^{n+1}}{e^n}
\]

\[
= (n+1) \left[ n! - \frac{n^n}{e^{n-1}} + \frac{n^n}{e^{n-1}} - \frac{(n+1)^n}{e^n} \right]
\]

\[
= (n+1) \left[ s_n + \left( \frac{n}{e} \right)^n \left( e - \left(1 + \frac{1}{n} \right)^n \right) \right],
\]

but

\[
\left( 1 + \frac{1}{n} \right)^n = 1 + \frac{1}{n} + \frac{n(n-1)}{2!} \left( \frac{1}{n} \right)^2 + \cdots + \frac{n!}{n!} \left( \frac{1}{n} \right)^n
\]

\[
\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e.
\]

Hence,

\[
s_{n+1} > (n+1)s_n.
\]

Also \( s_n \geq 0 \) for all \( n \); we can prove this inductively. When \( n = 1, s_1 = 0 \). Suppose that \( n! \geq n^n/e^{n-1} \) for some \( n \). Then

\[
(n+1)! \geq \frac{(n+1)^{n+1}}{e^n}
\]

\[
= \frac{(n+1)^n}{e^n} \left( 1 + \frac{1}{n+1} \right)^n > \frac{(n+1)^n}{e^n}.
\]

Hence, \( s_{n+1} > (n+1)s_n \) for all \( n \). Also, for all \( n \geq 2 \), \( s_{n+1} > (n+1)s_n \geq (n+1)s_2 \) and \( s_2 > 2s_1 = 0 \), so the sequence \( (s_n) \) is unbounded.
**Reviews**


The blurb on the back of this book claims that the book ‘introduces fundamental ideas in Mathematics through interesting puzzles’, its motivation being that ‘many bright students … are bored with routine class work … and will enjoy these puzzles which will sharpen their logical reasoning’. It certainly succeeds, being engaging to the reader, and offering a series of exercises highlighting certain key ideas inherent in many forms of problem solving. Its range of topics is as diverse as it is entertaining, from noughts and crosses to the famous bridges of Königsberg, and from warfare to Gödel’s theorem. There are sixteen chapters each dealing with a different and interesting facet of mathematics of interest to this book’s target age group of 12–18 year olds.

I would unreservedly recommend this book to anyone at all interested in mathematics, as its refreshing approach to the topic of problem solving is a welcome break from the stodgy fare of the stipulated syllabi in schools. Indeed, in addition to being a useful stimulus to 12–18 year olds, it could also be of interest to many adults, both as an aid to understanding, and as a source of new ideas.

Student, Berkhamsted Collegiate School  
Paul Jefferys


If you regard mathematicians as a rather eccentric breed, then you will be confirmed in this view by reading this book. It is not a book about mathematics, but about mathematicians, a cornucopia of their idiosyncrasies. There is Dirichlet, who made an exception to his rule of never writing letters when his first child was born: he wired his father-in-law with the message ‘2 + 1 = 3’. Or Dedekind, who read his own obituary in a publication of the German Mathematical Society, giving the day, month and year of his death; he wrote to the editor: ‘at least the year is wrong’. Or Gauss, who, deep in mathematical thought, received a report that his wife was dying. ‘Tell her to wait a few minutes until I have finished’, was his reply. Then there is an article that Littlewood wrote which ended with the statement ‘Thus σ should be made as small as possible’. When it appeared in print, the statement was omitted; instead a speck appeared which turned out to be the tiniest σ anyone had ever seen.

This book is full of such stories. The centre of gravity is the tiniest σ in the history of mathematics. The book consists for the most part of a catalogue of ‘hinged’ dissections, arranged according to the techniques used for discovering them and the figures involved. The techniques are fascinating due to their surprising use of symmetry and tessellation. The sheer weight of examples means that this is a book squarely aimed at the enthusiast; there is an enormous amount of material here, and it requires some passion to wade through it. But there is variety also: different types of hinges are flirted with, and a number of striking three-dimensional dissections are given. There are also exercises, complete with answers. A detective story (‘The Curious Case of the Brass Hinges’) unfolds in short chapters throughout the book, trying to get to the bottom of an apparent mathematical cover-up instigated by Henry Dudeney, a puzzle-column writer from the turn of the last century.

Frederickson’s book is full of gorgeous results, described in lively style; it will reward serious study, and provide a massive store of puzzles.

Queens’ College, Cambridge  
Will Donovan


Instead of taking the textbook approach to teaching maths, this book tries to make the subject more accessible by presenting topics in an encyclopaedia format, from ‘absolute value’ to ‘Zeno’s paradox’. This means that rather than having to wade through a whole branch of maths to find out about a single area, you can simply look it up. For example, instead of studying statistics until you understand what lies behind combinations and permutations, you can find it under ‘C’ in The A to Z of Mathematics.

The book contains three types of article: explanations (such as ‘algebra’), definitions (‘proportion’), and other items worth mentioning (‘hexomino’, or the ‘four colour problem’). Rather than just telling the reader about the subjects, it explains each in enough depth to be able to give some
understanding of the concepts; and articles often include worked examples to demonstrate. For the more common subjects, such as ‘algebra’ or ‘probability’, there are in-depth explanations with plenty of examples showing different variations on the topic, whereas for smaller subjects, such as ‘factorial’ there is just a simple explanation and example. However, explanations generally go into enough depth to leave the reader with enough knowledge to understand the basics of the subject.

The author states that the book is aimed at ‘the millions of people who would love to understand math but are turned away by fear of its complexity’, and in general articles start from a basic enough level to avoid scaring off readers. Of course, not all articles can work from an assumption of no background knowledge — many articles state algebraic formulae or use trigonometry, but no subject is used that isn’t covered elsewhere in the book, and related areas are usually referenced at the end of an article. Overall, the book lives up to its title, and could be useful to anyone, whether they’re trying to pick up the basics or a maths student looking for a quick reference book.

Student, Paston College, Norfolk

Stephen Walker

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ISSN 0025-5653